

Graph Theory

Part Three

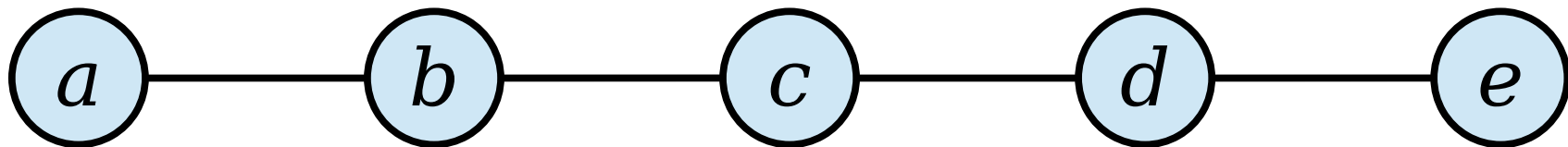
Outline for Today

- ***The Pigeonhole Principle***
 - A simple yet surprisingly effective fact.
- ***Graph Theory Party Tricks***
 - Cool tricks to try at your next group meeting.
- ***A Little Movie Puzzle***
 - Who watched what?

Recap from Last Time

Adjacency and Reachability

- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***reachable*** if there's a path between them.



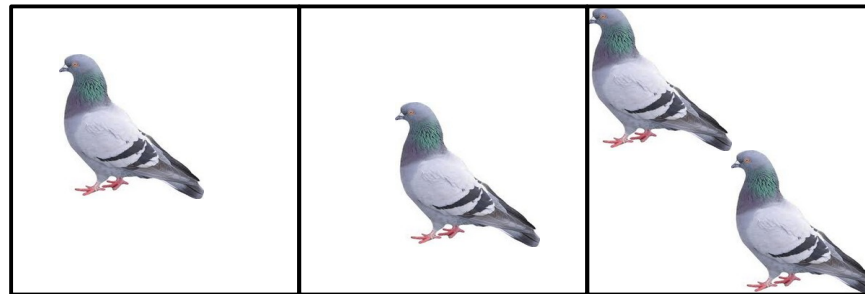
New Stuff!

The Pigeonhole Principle

The Pigeonhole Principle

Theorem (The Pigeonhole Principle):

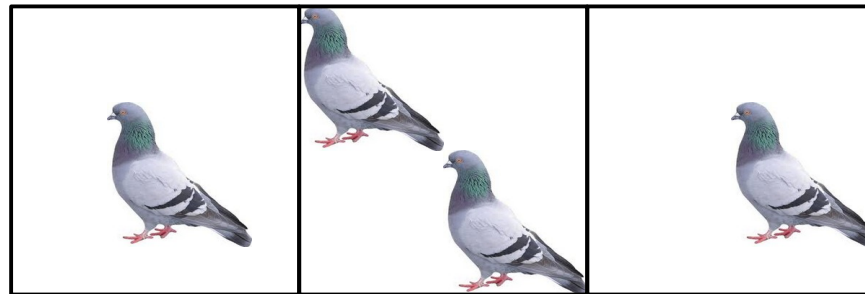
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



The Pigeonhole Principle

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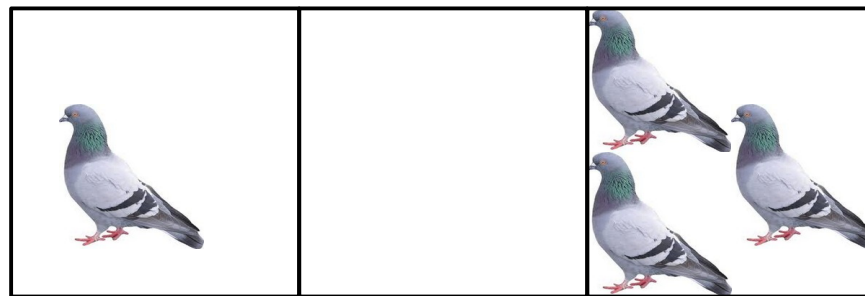
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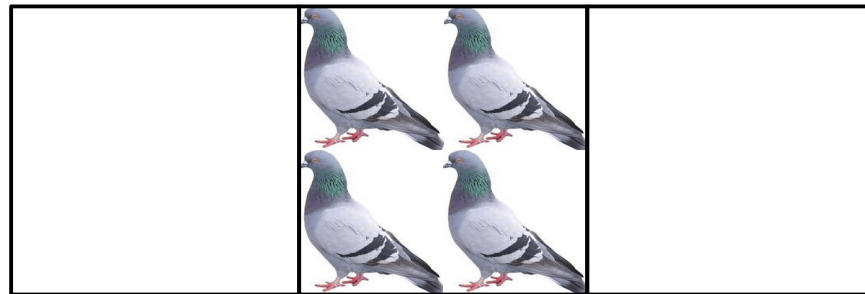
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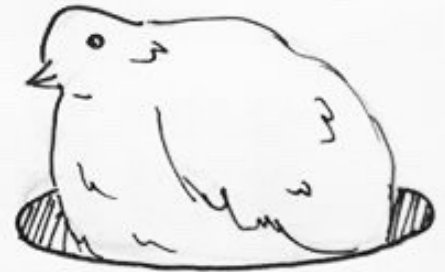
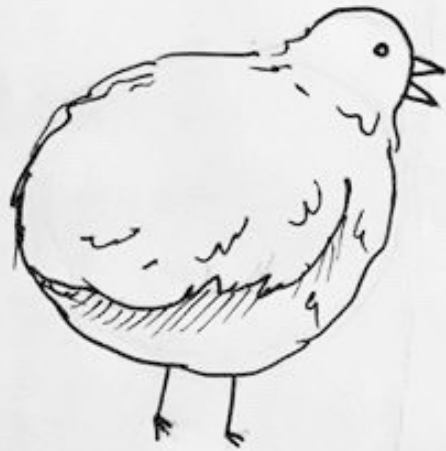
The Pigeonhole Principle

Theorem (The Pigeonhole Principle):

If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



NO MORE
- PIGEON HOLES?!



$$m = 4, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
 - 366 possible birthdays (pigeonholes).
 - 367 people (pigeons).
- Two people in San Francisco have the exact same number of hairs on their head.
 - Maximum number of hairs ever found on a human head is no greater than 500,000.
 - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. How many of the following statements are true?

- (1) If $f : A \rightarrow B$, then f is injective.
- (2) If $f : A \rightarrow B$, then f is not injective.
- (3) If $f : A \rightarrow B$, then f is surjective.
- (4) If $f : A \rightarrow B$, then f is not surjective.

Answer at

<https://pollev.com/cs103aut23>

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . There are m objects in total, so we know that

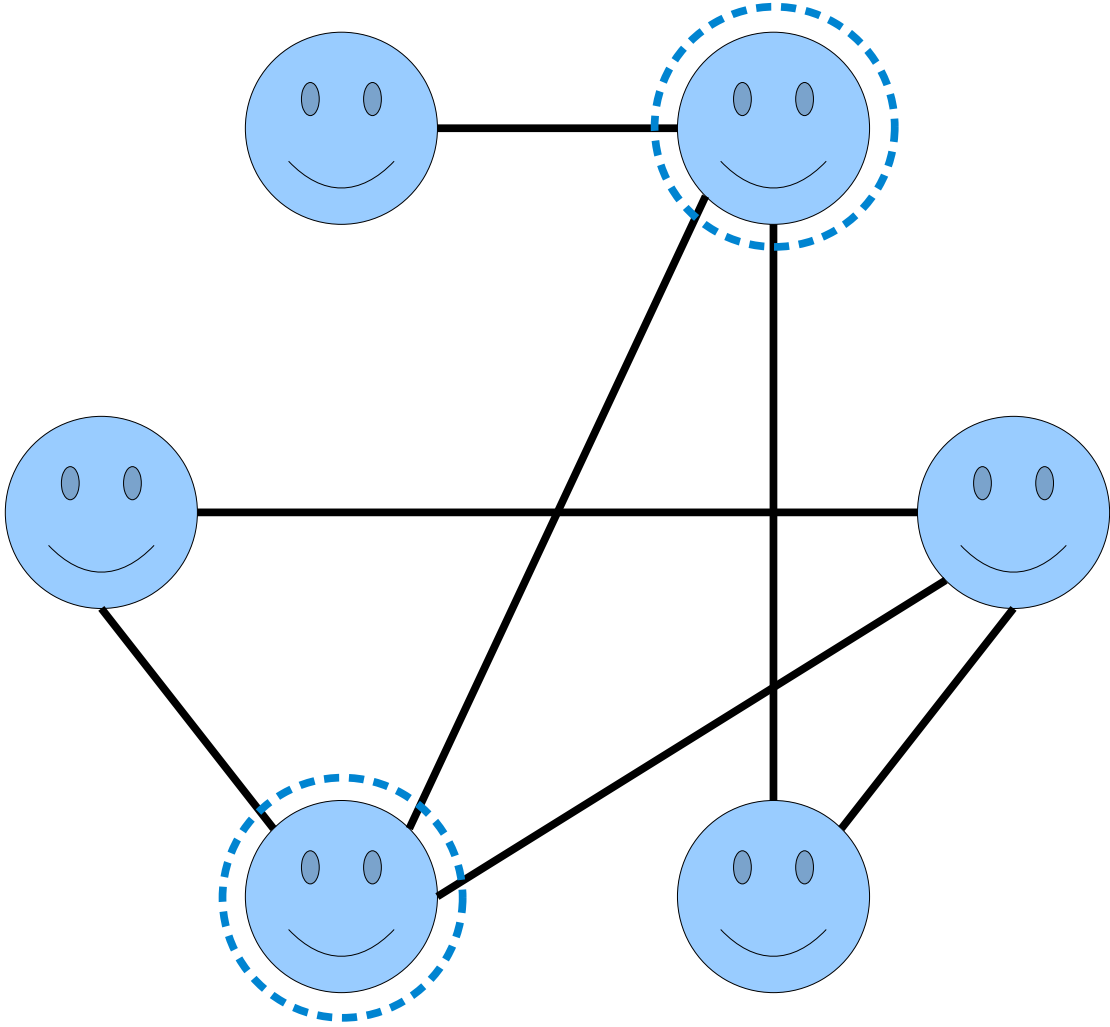
$$m = x_1 + x_2 + \dots + x_n.$$

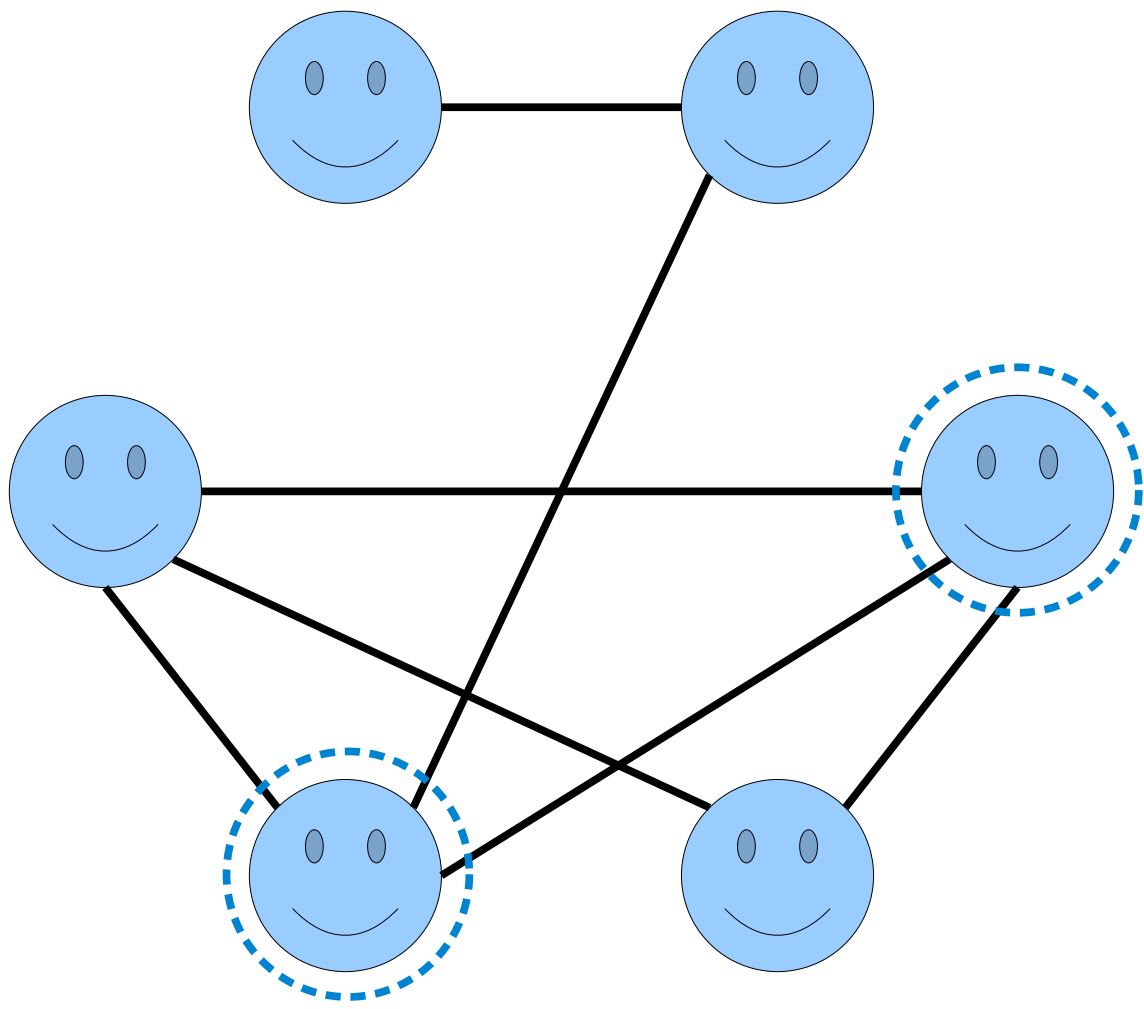
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i . This means that

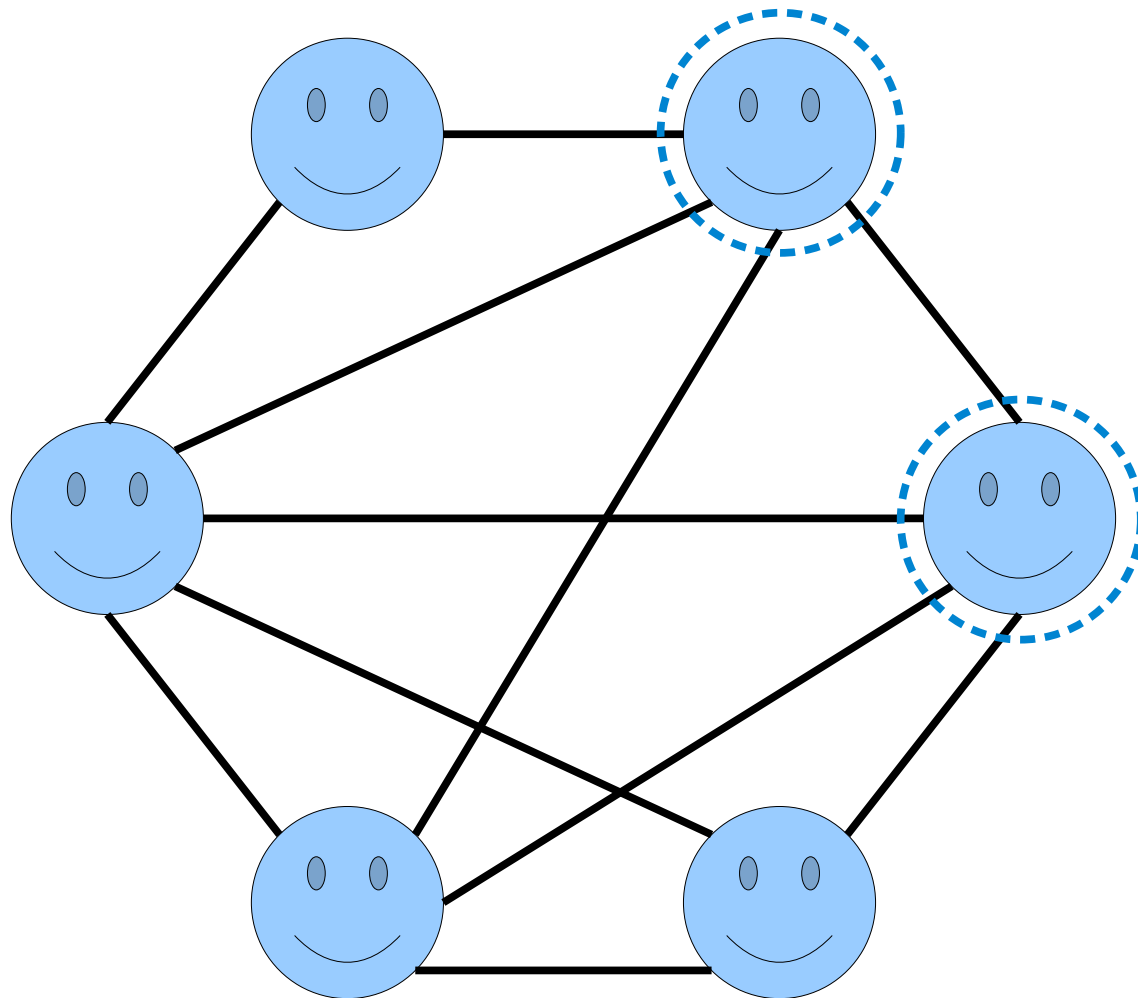
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &\leq 1 + 1 + \dots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

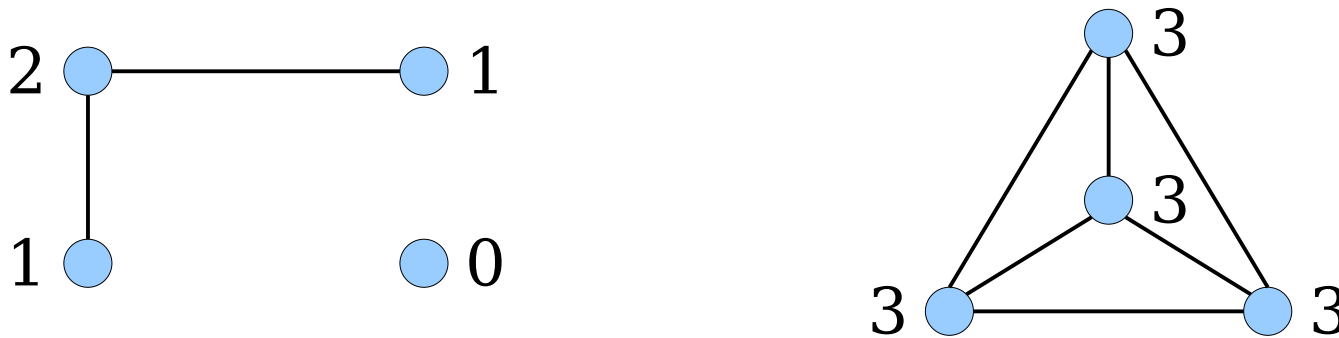




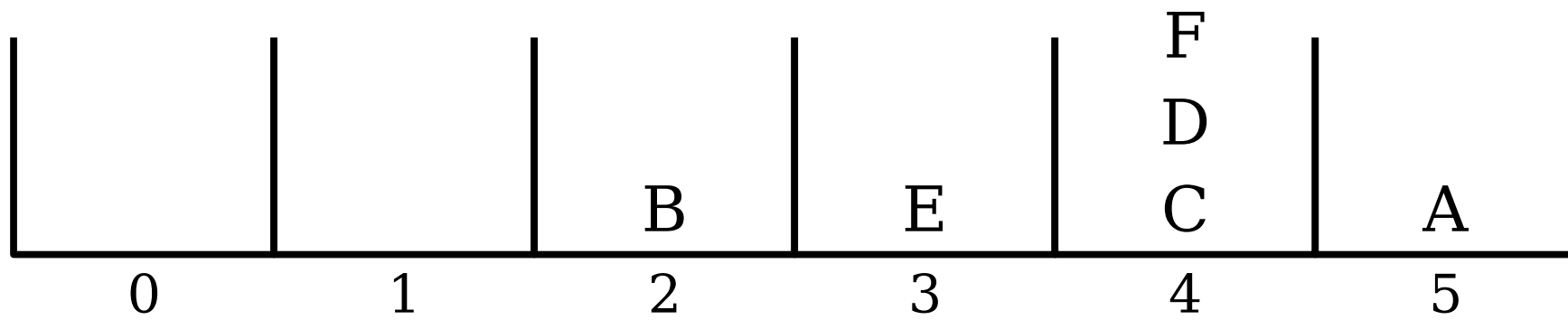
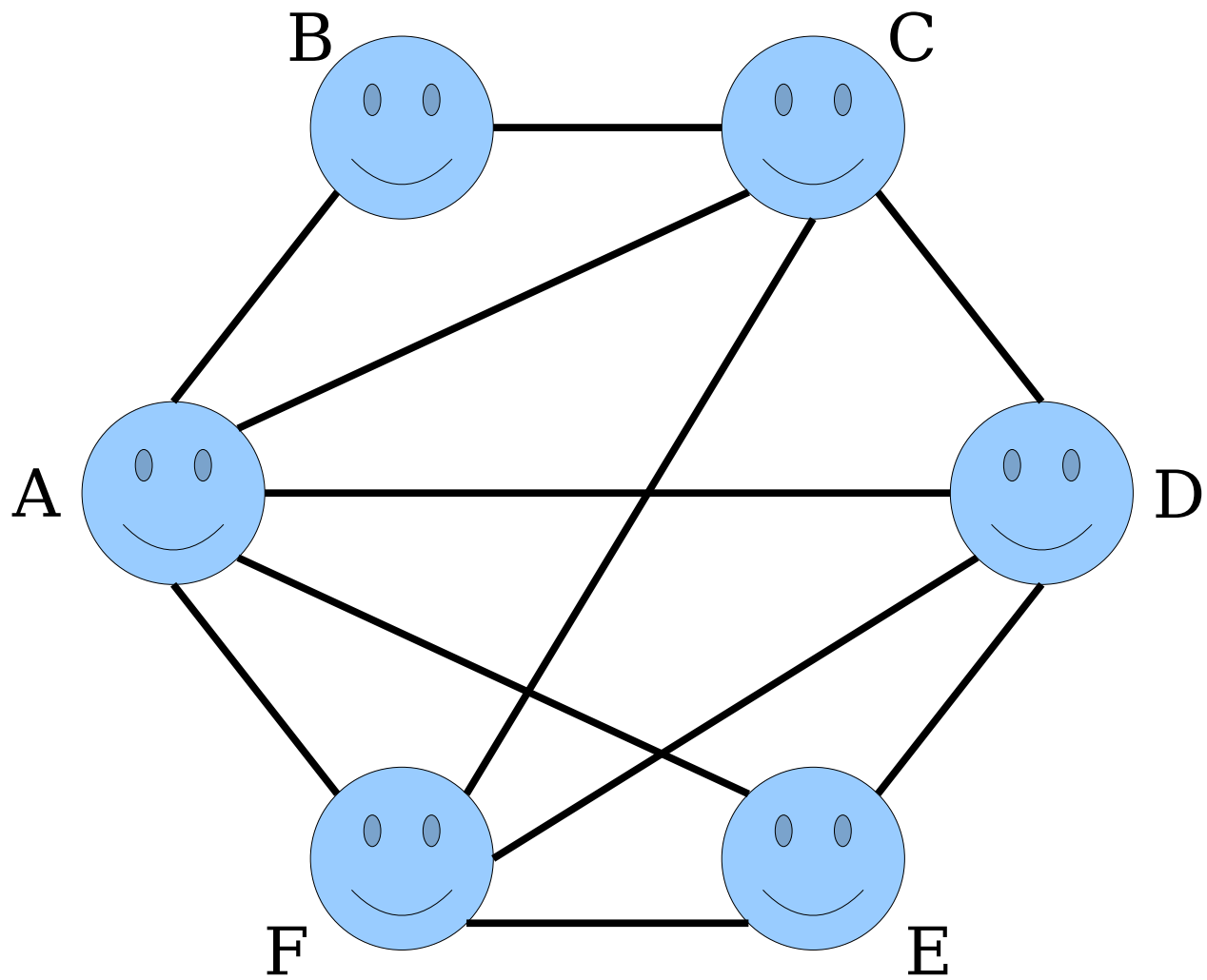


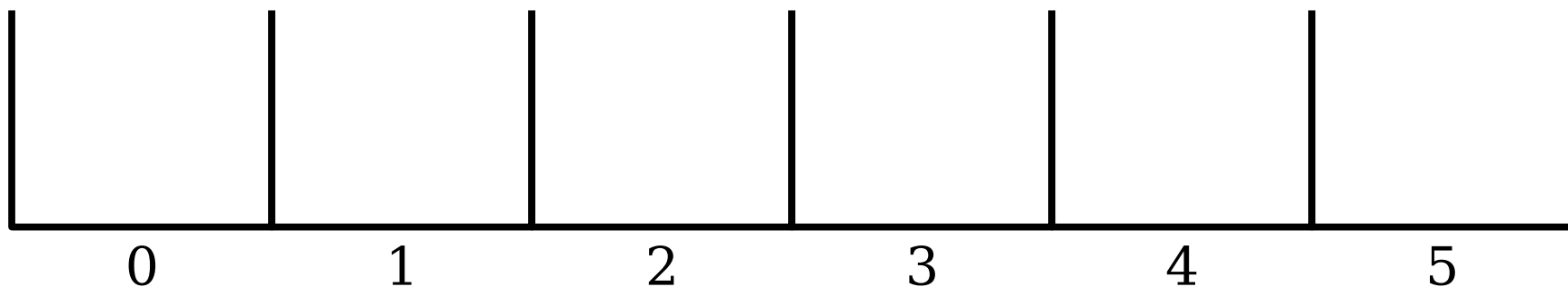
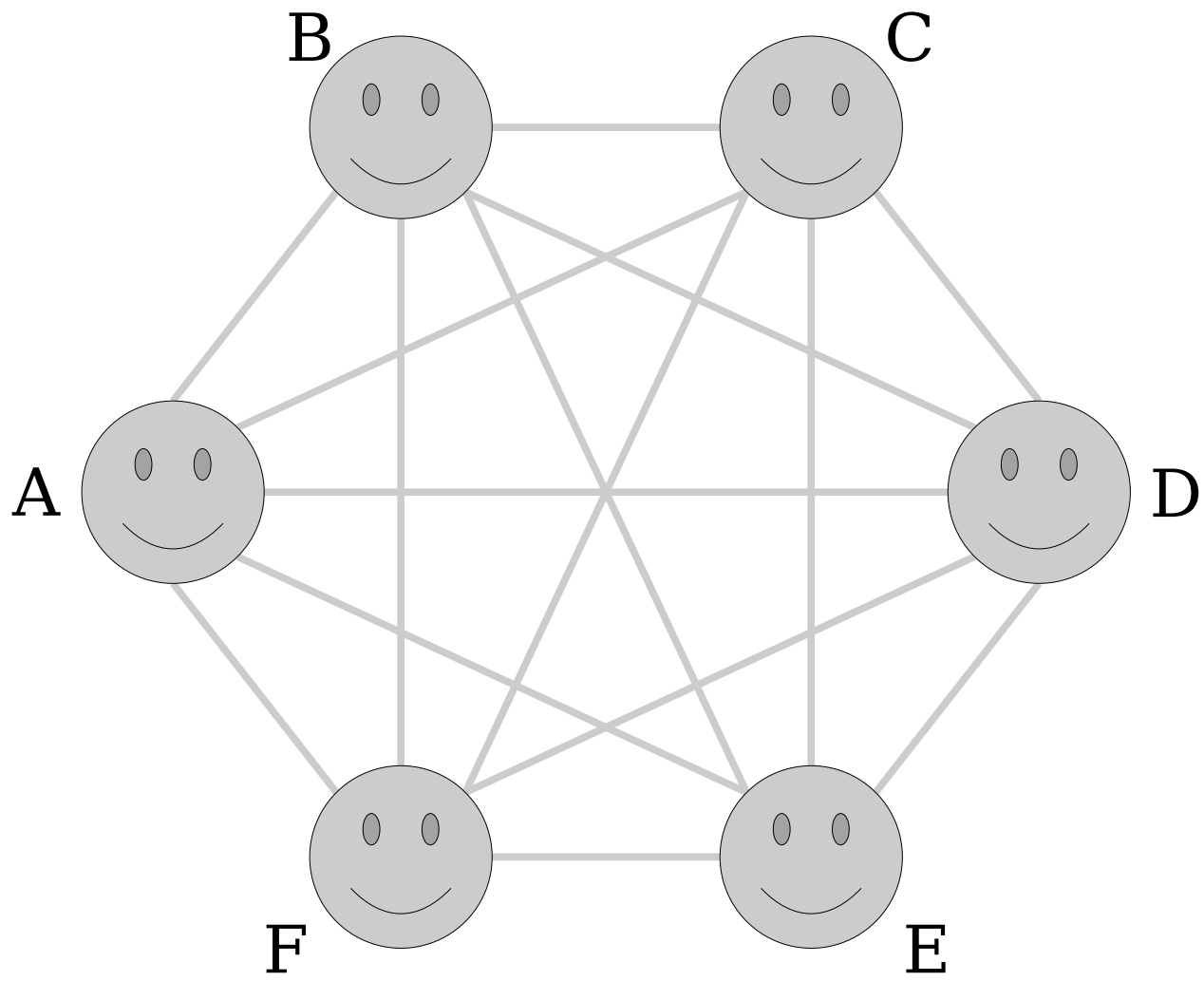
Degrees

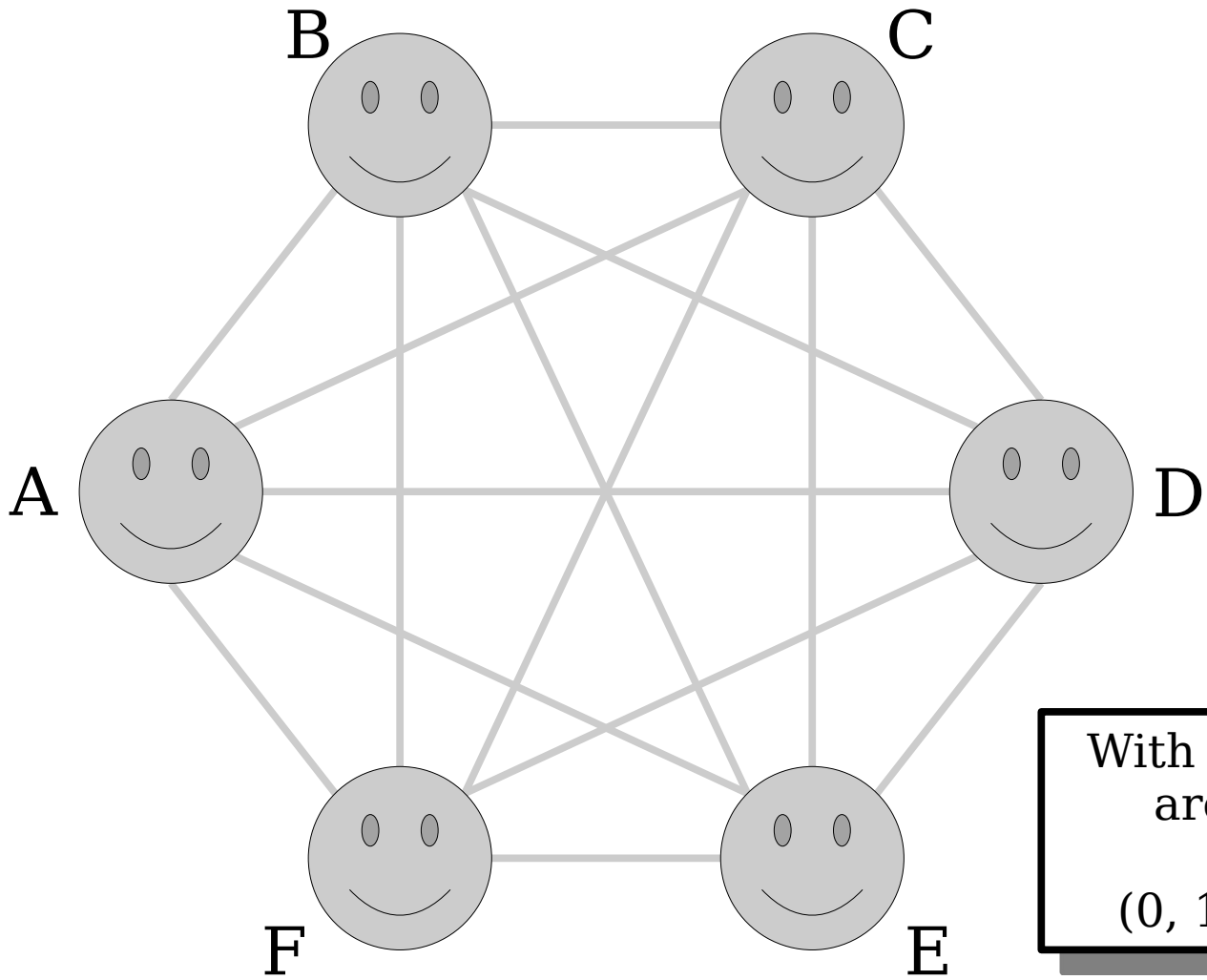
- The **degree** of a node v in a graph is the number of nodes that v is adjacent to.



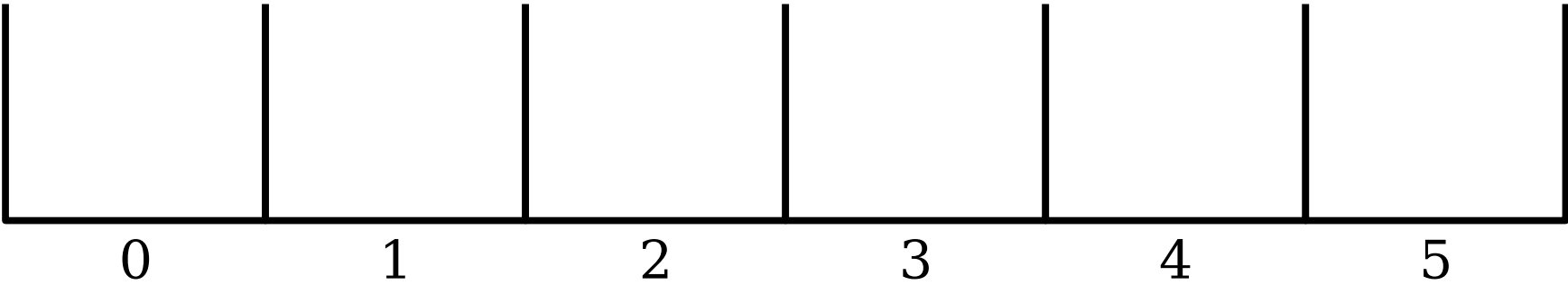
- Theorem:** Every graph with at least two nodes has at least two nodes with the same degree.
 - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.

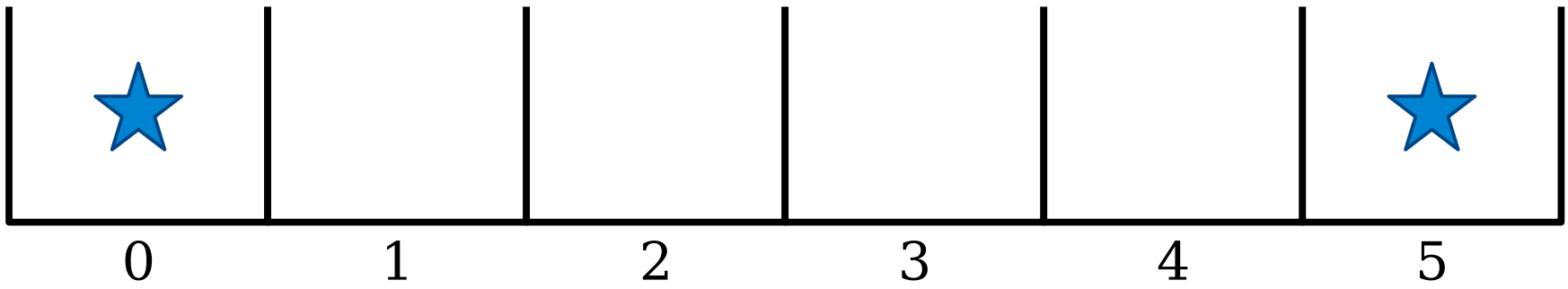
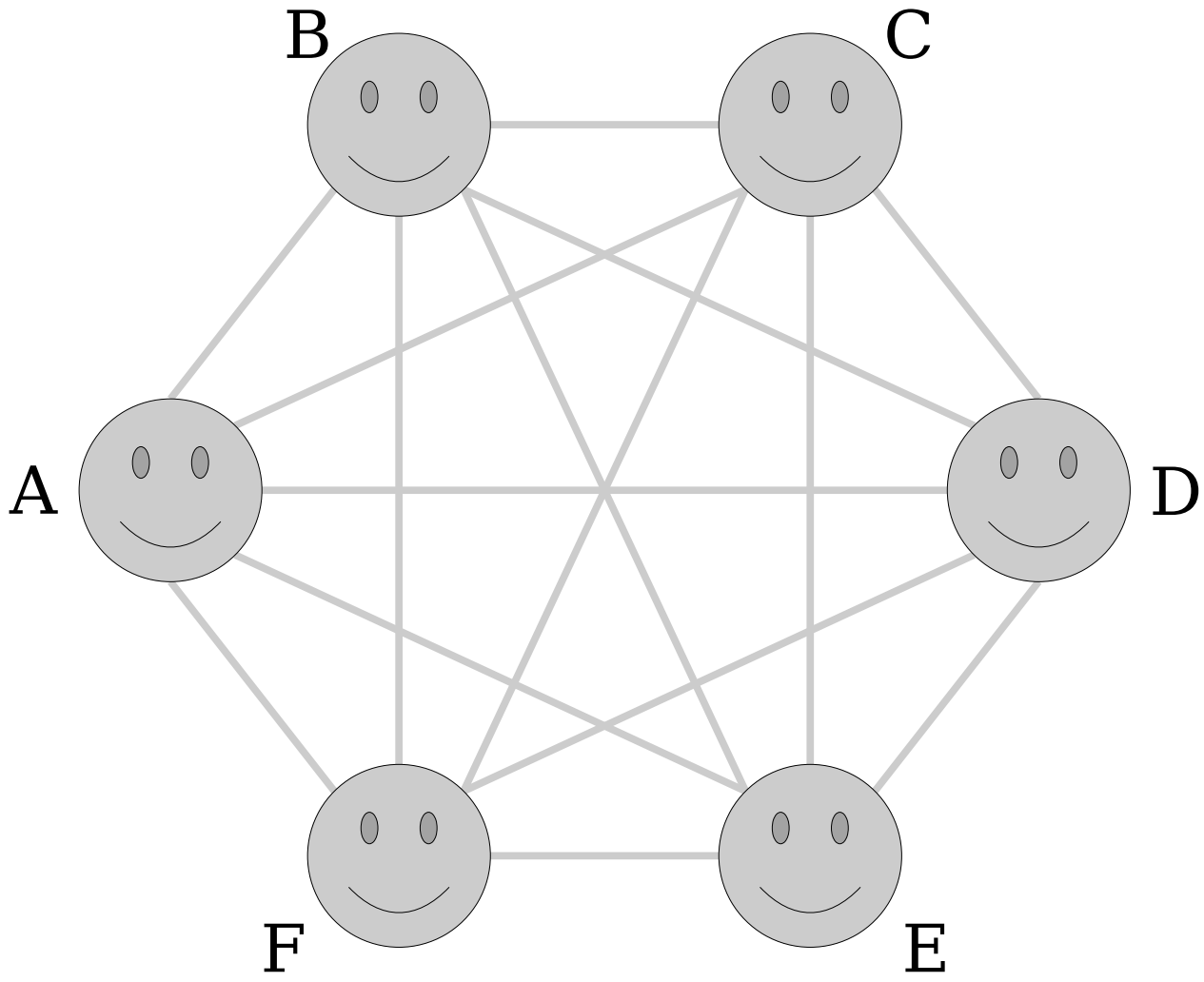


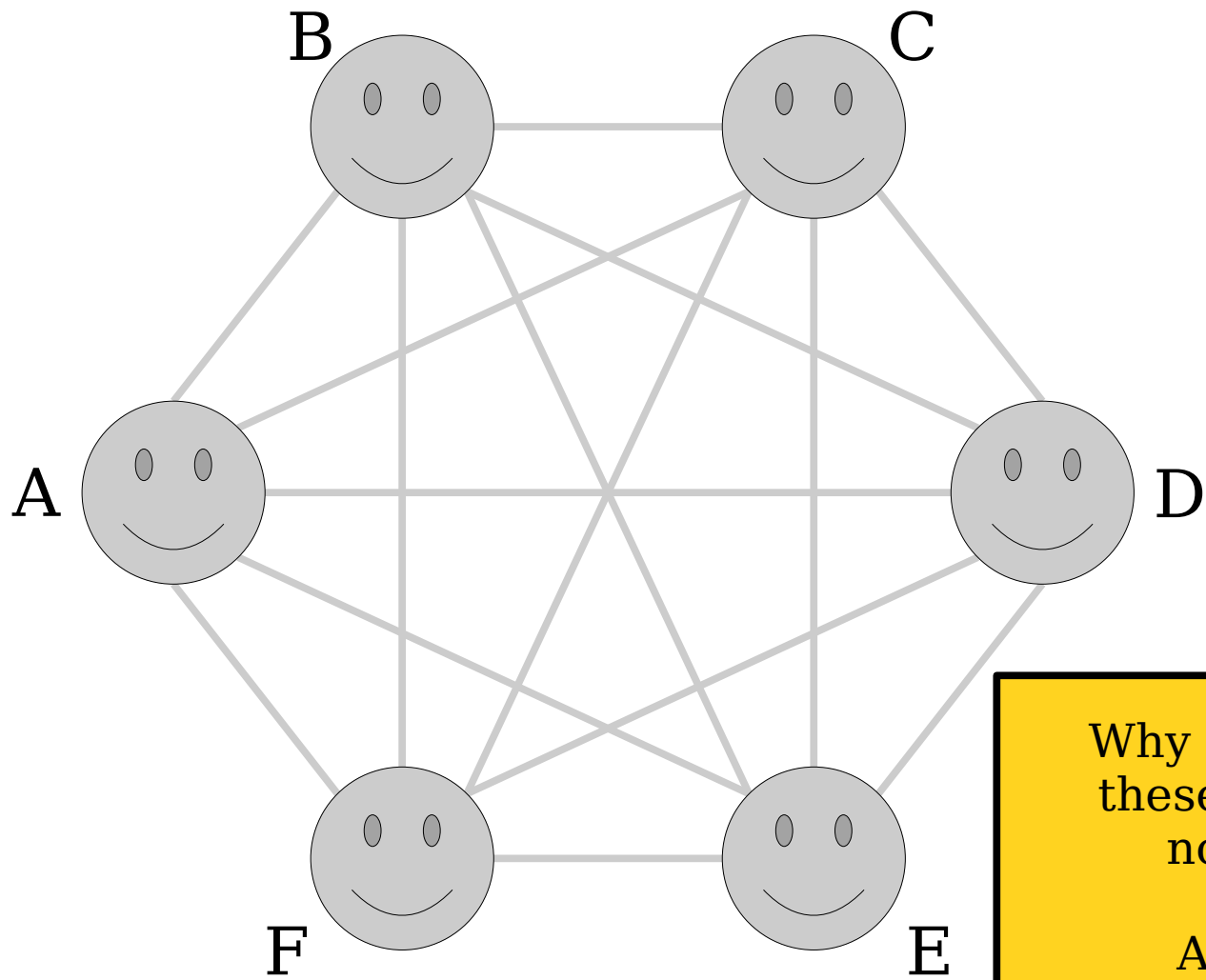




With n nodes, there are n possible degrees
(0, 1, 2, ..., $n - 1$)

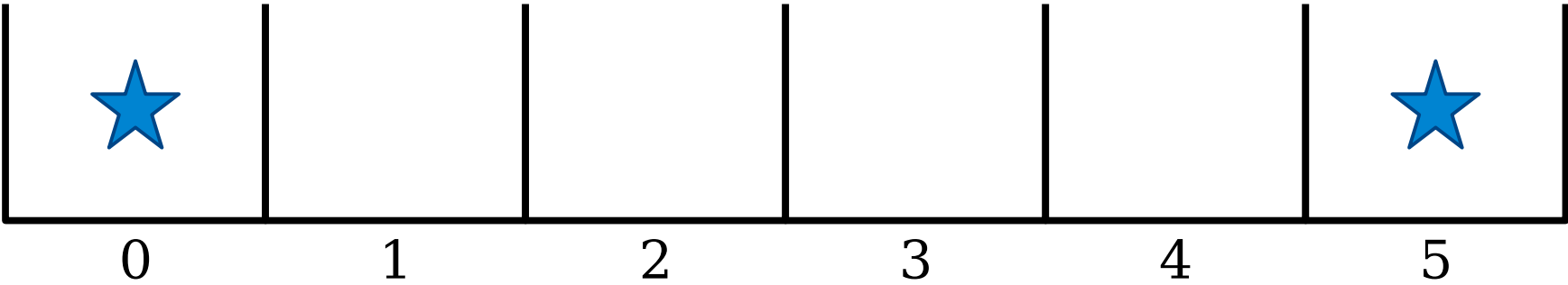


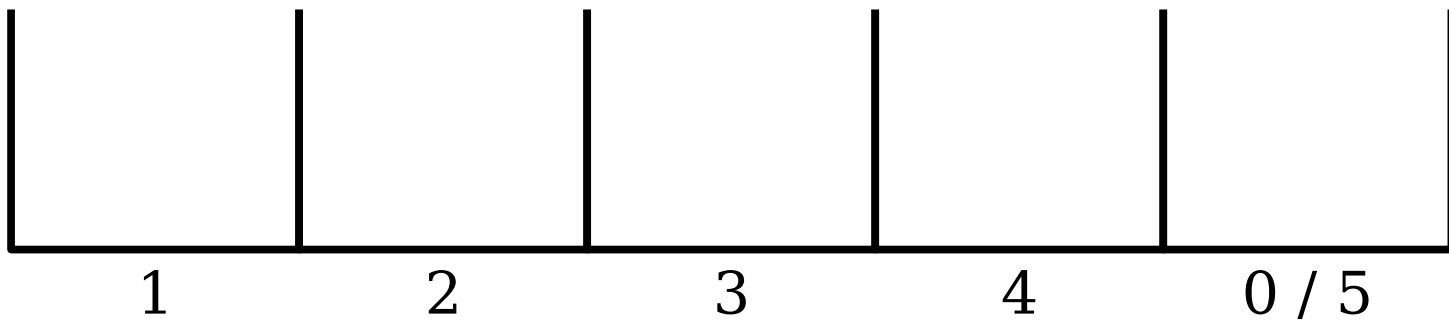
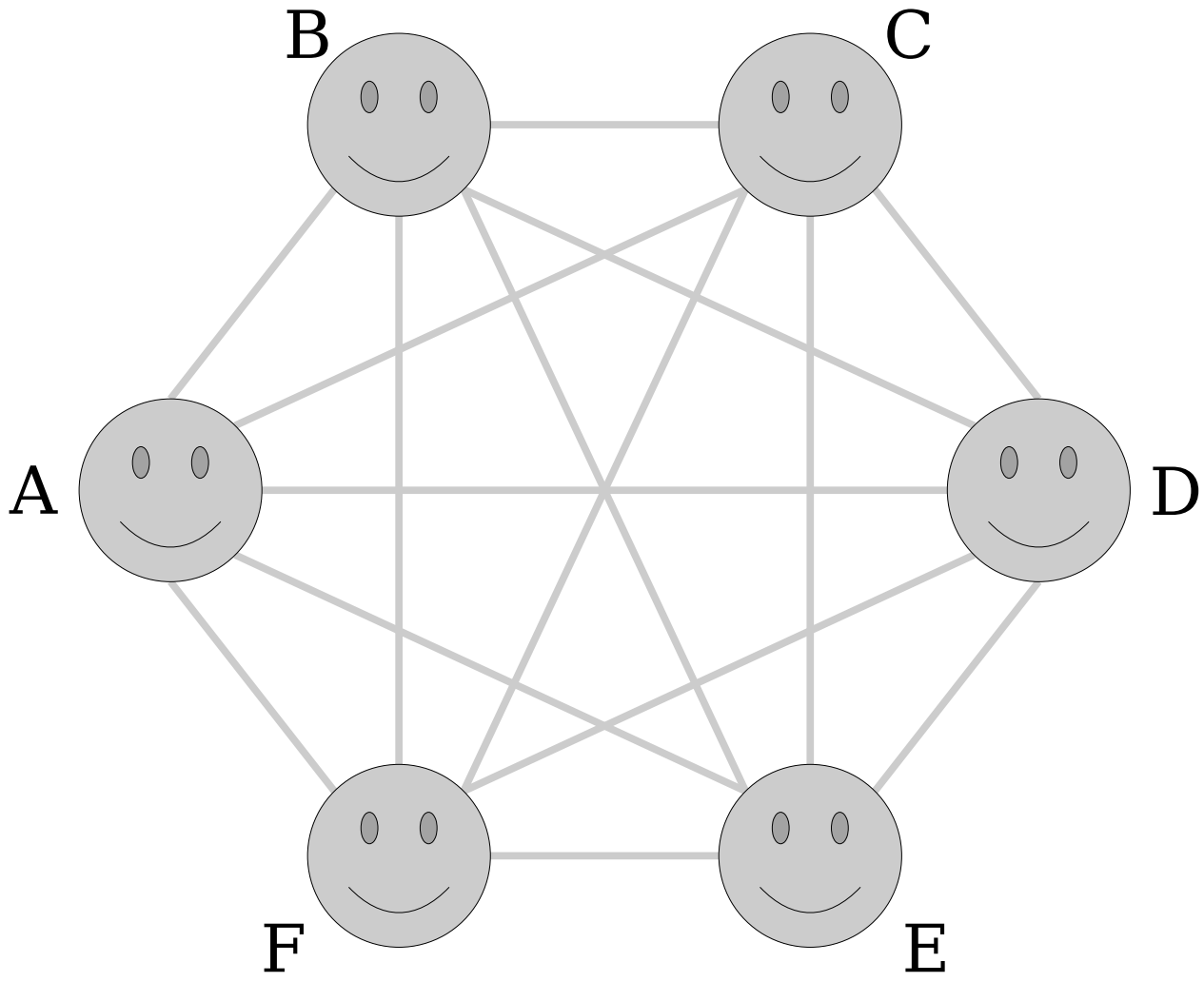




Why can't both of these buckets be nonempty?

Answer at pollev.com/cs103aut23





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We claim that G cannot simultaneously have a node u of degree 0 and a node v of degree $n - 1$:

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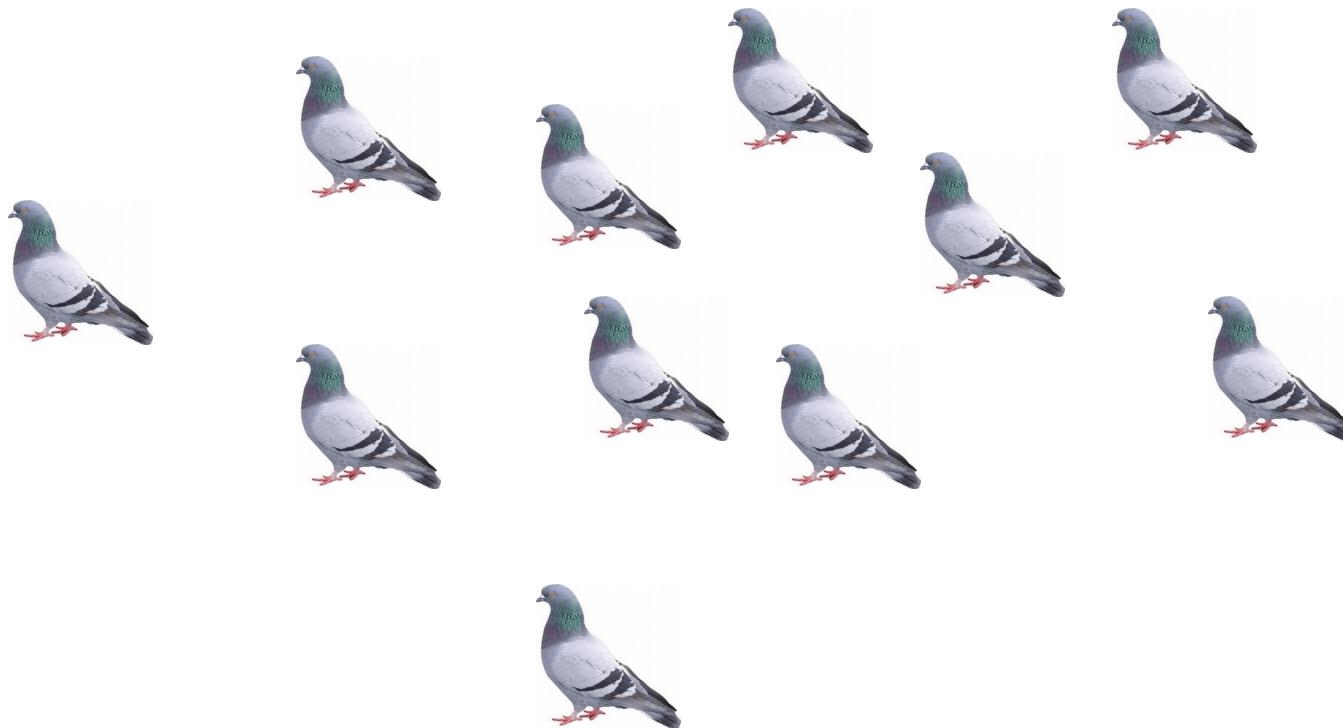
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G , namely $0, 1, 2, \dots, n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

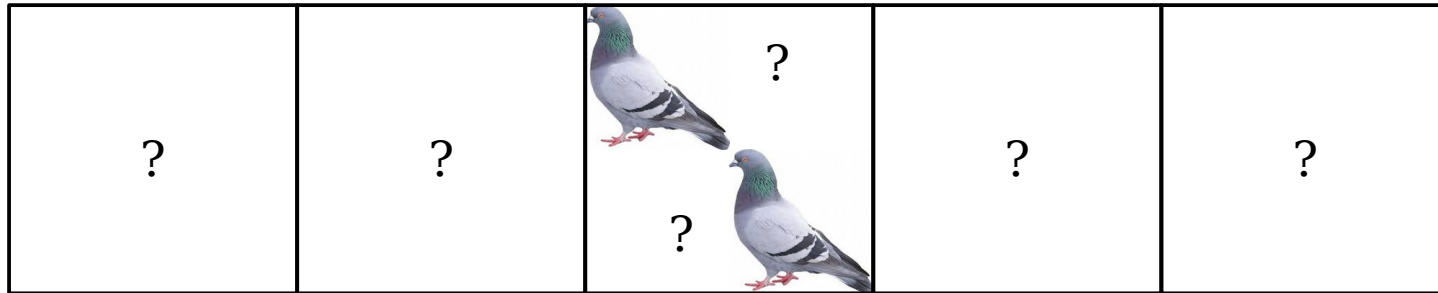
We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

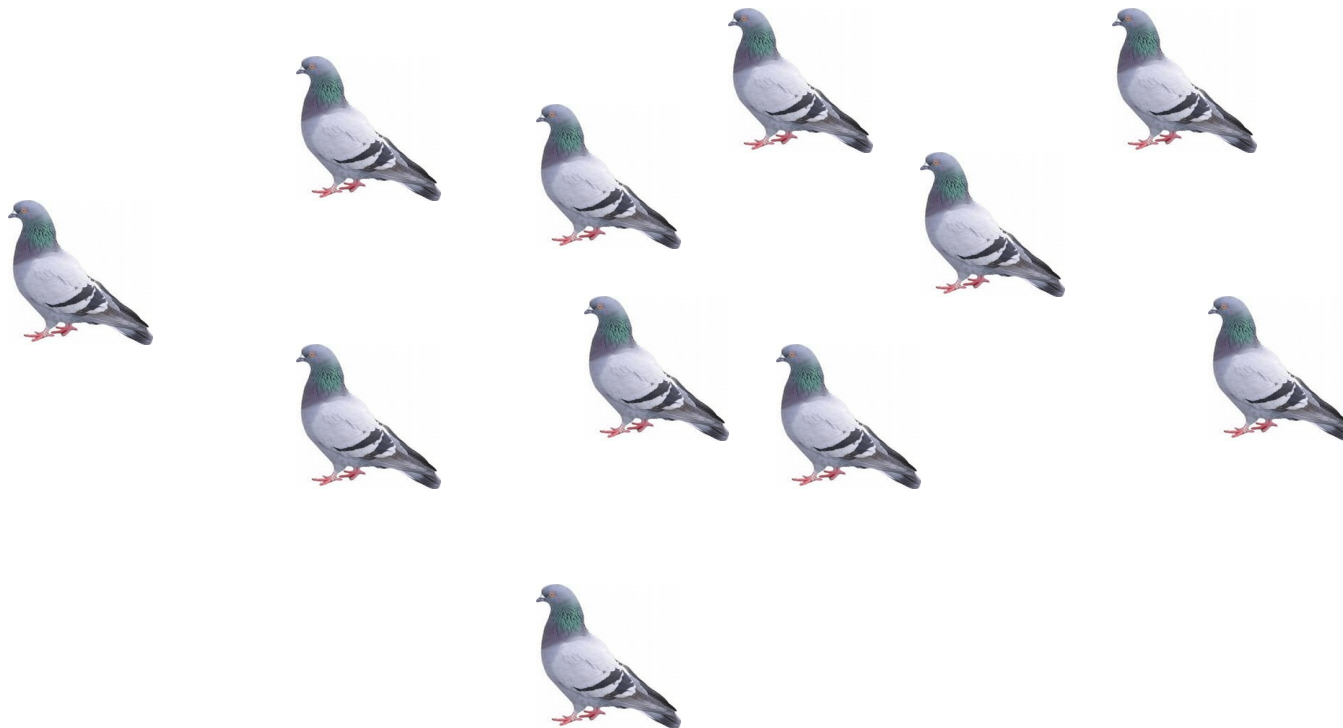
The Pigeonhole Principle



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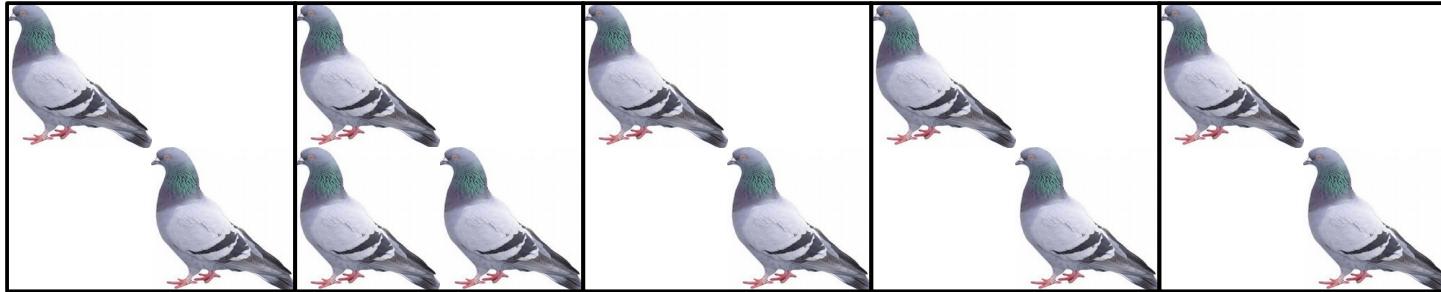
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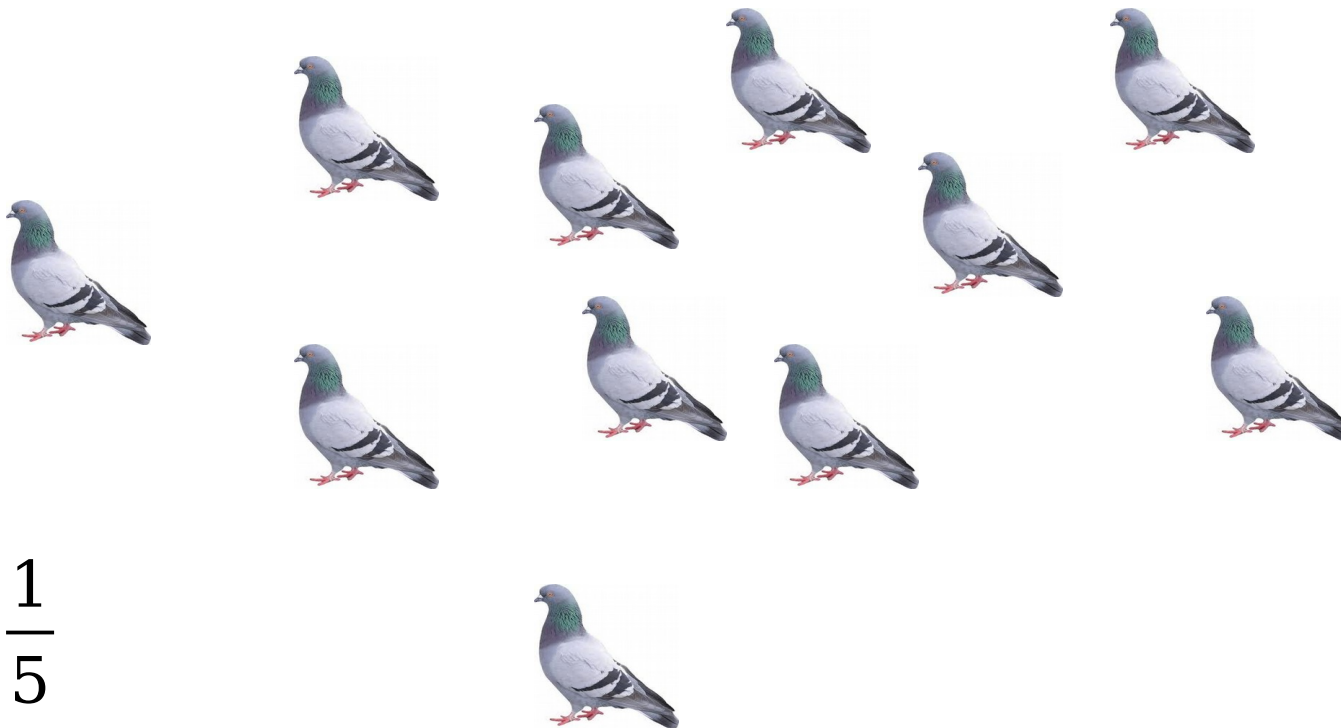
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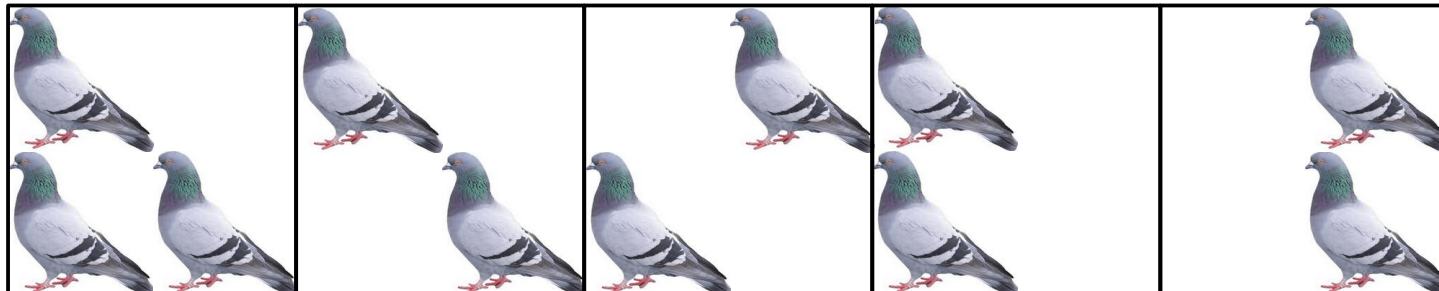


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The **generalized pigeonhole principle** says that if you distribute m objects into n bins, then
 - some bin will have at least $\lceil m/n \rceil$ objects in it, and
 - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means “ m/n , rounded up.”
 $\lfloor m/n \rfloor$ means “ m/n , rounded down.”



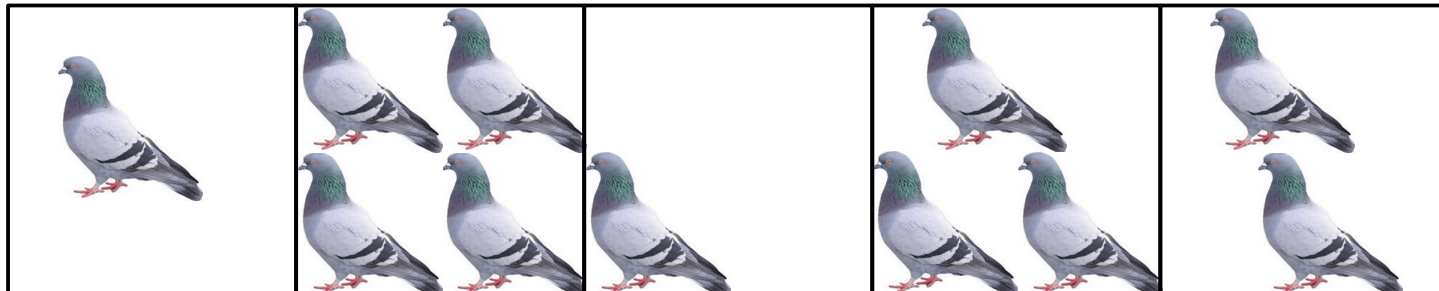
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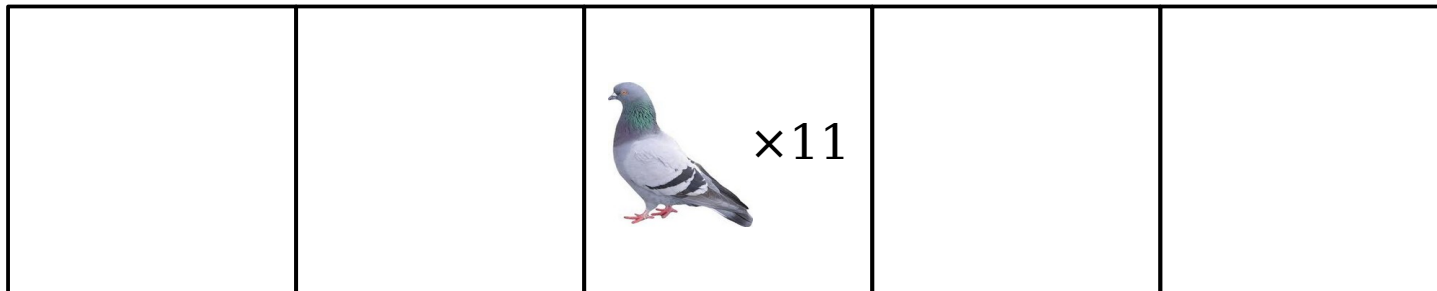
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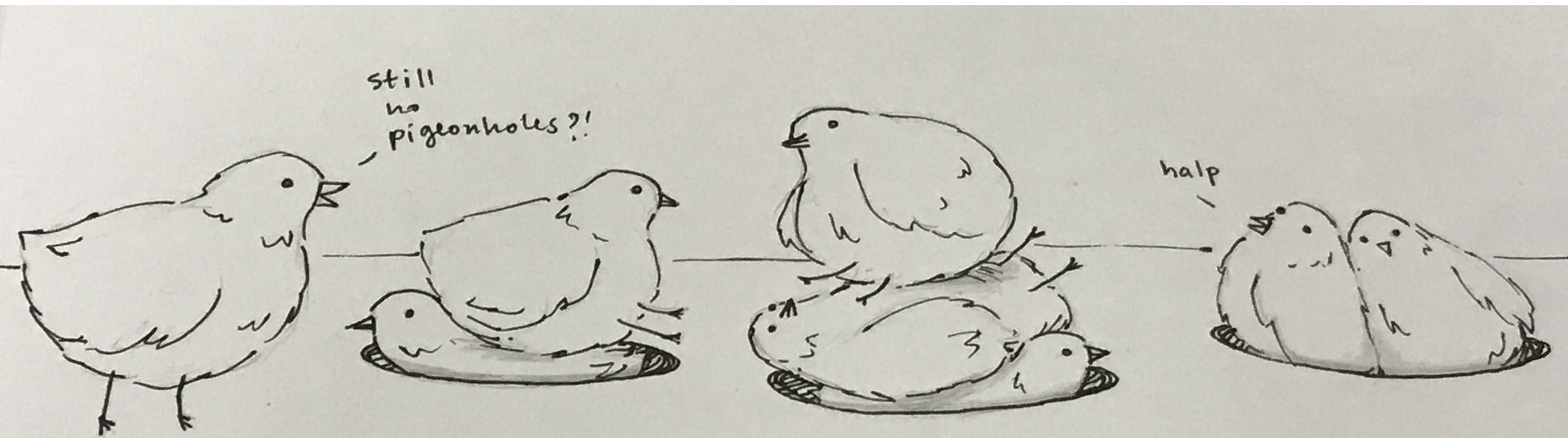
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$$m = 8, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n , there is a way to distribute m objects into n bins such that each bin contains fewer than $\lceil m/n \rceil$ objects.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . Since there are m objects in total, we know that

$$m = x_1 + x_2 + \dots + x_n.$$

Since each bin contains fewer than $\lceil m/n \rceil$ objects, we see that $x_i < \lceil m/n \rceil$ for each i . Therefore, we have that

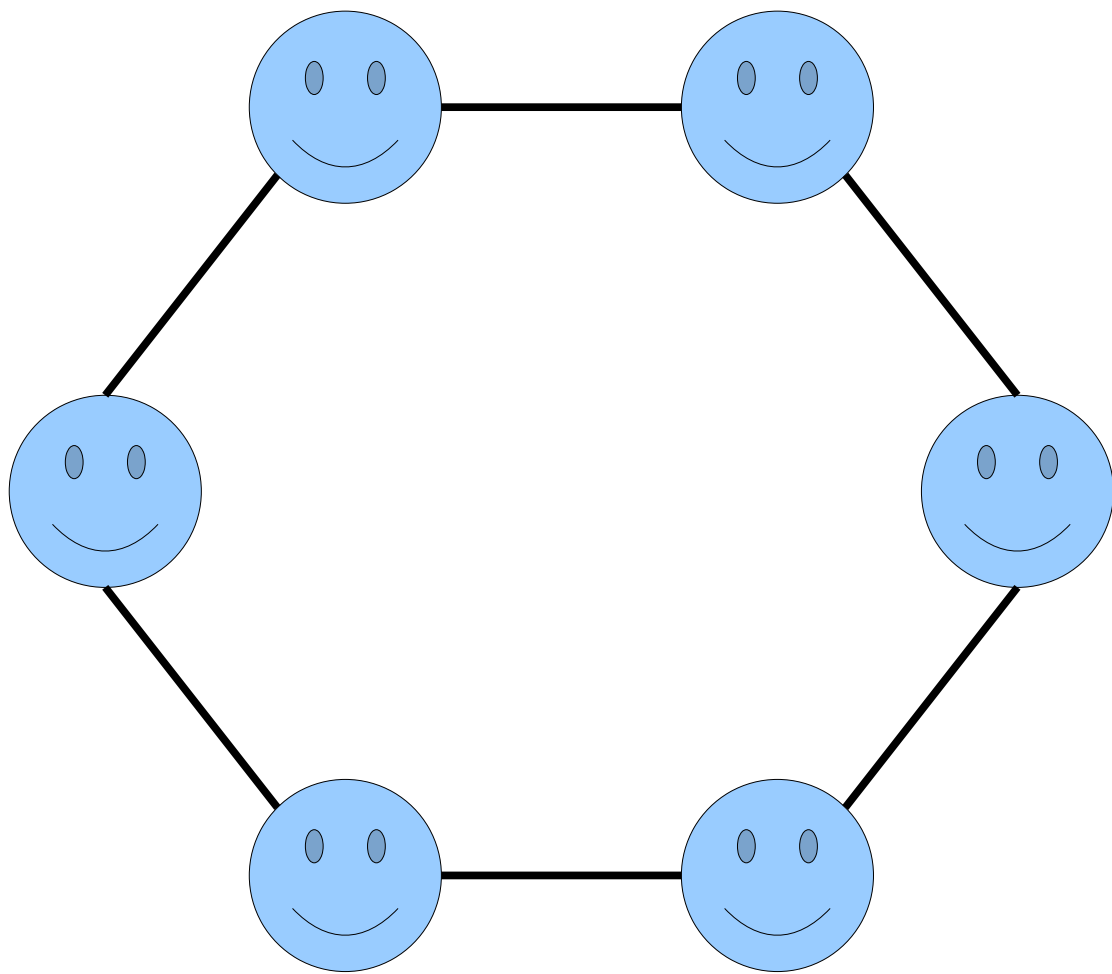
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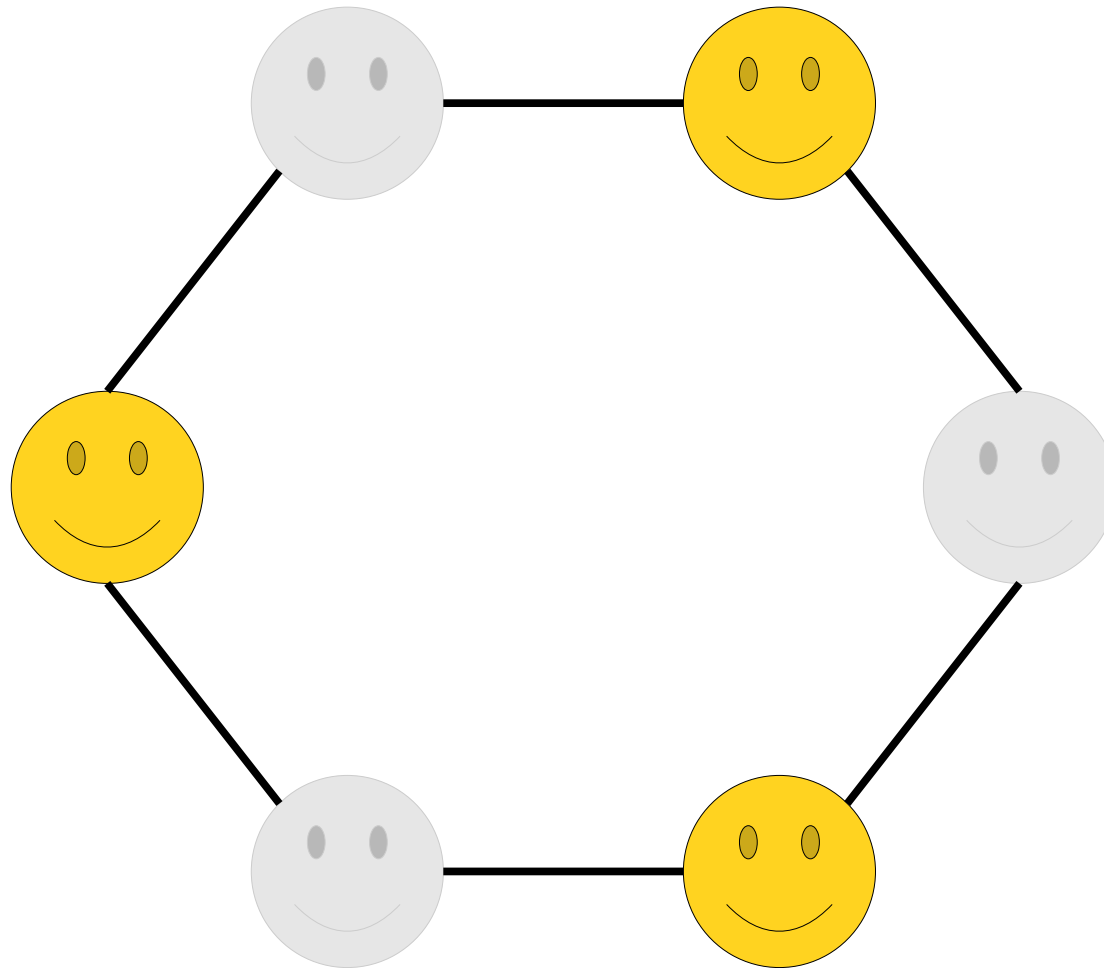
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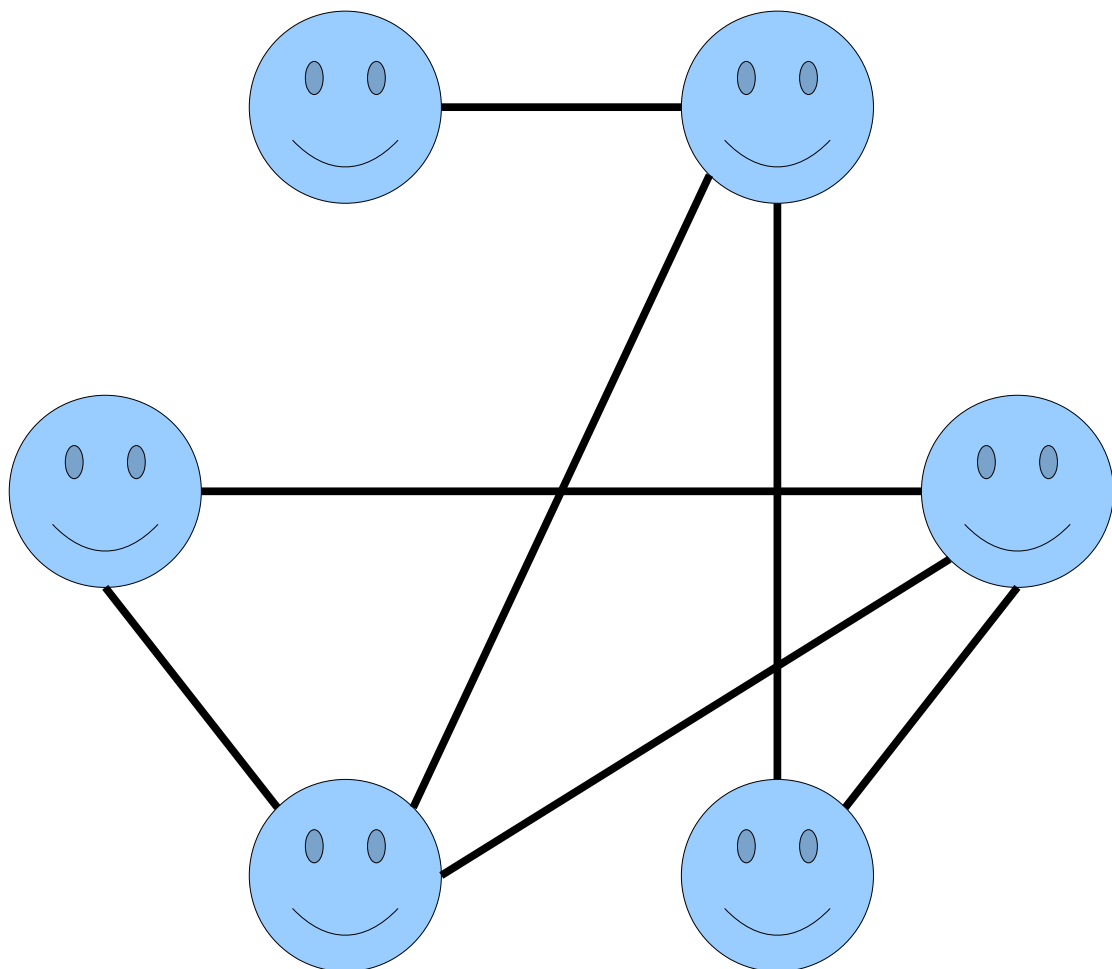
An Application: Friends and Strangers

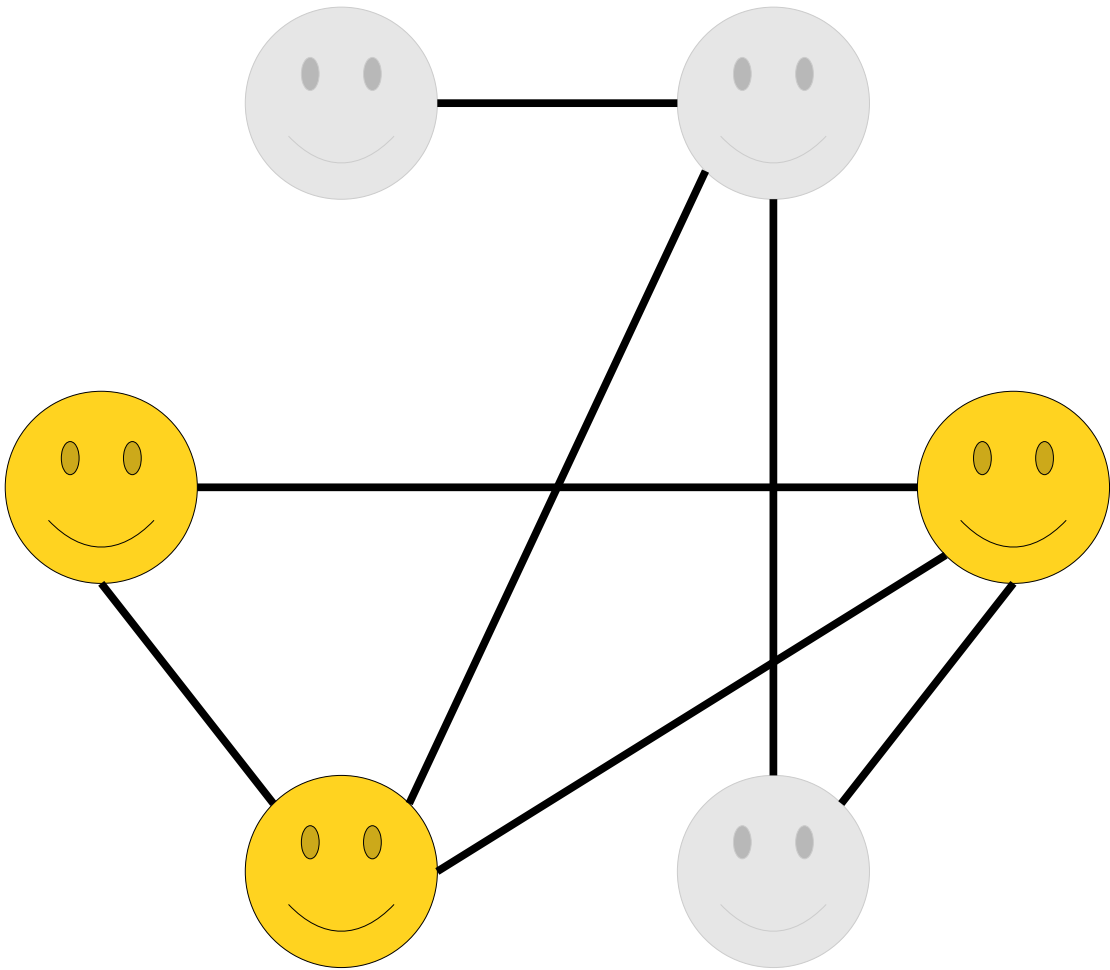
Friends and Strangers

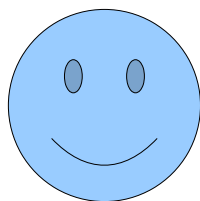
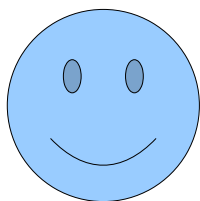
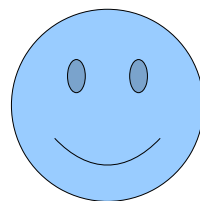
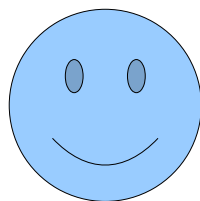
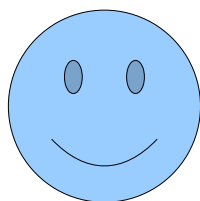
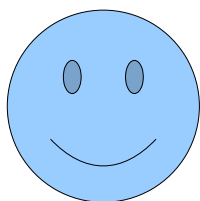
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

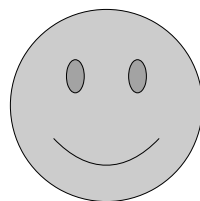
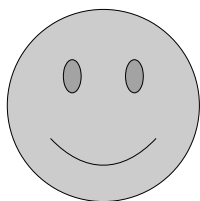
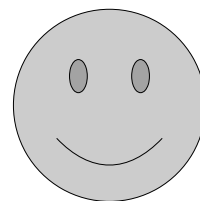
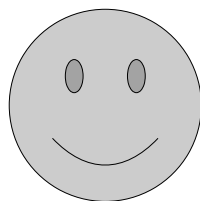
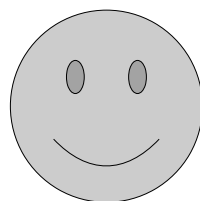
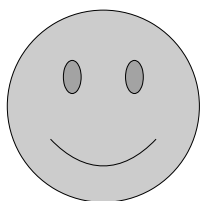


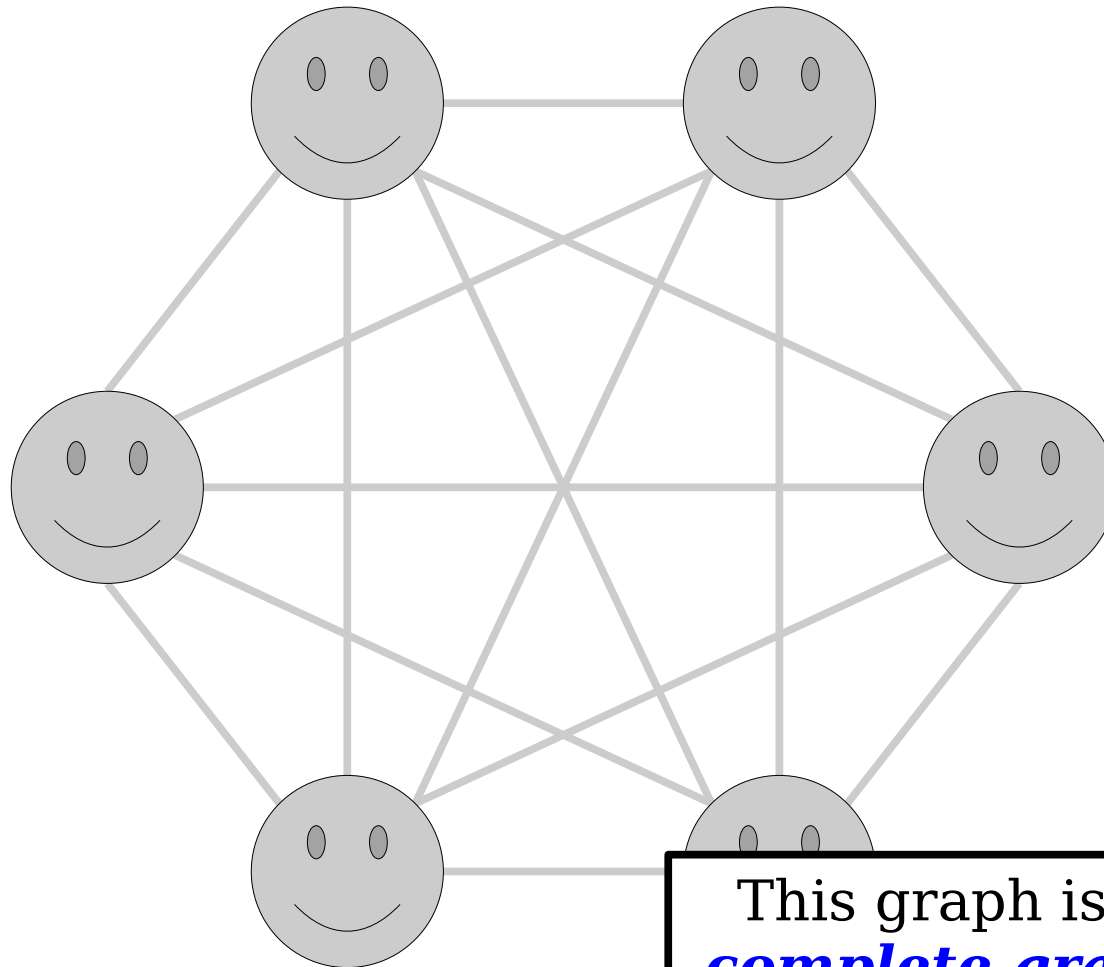




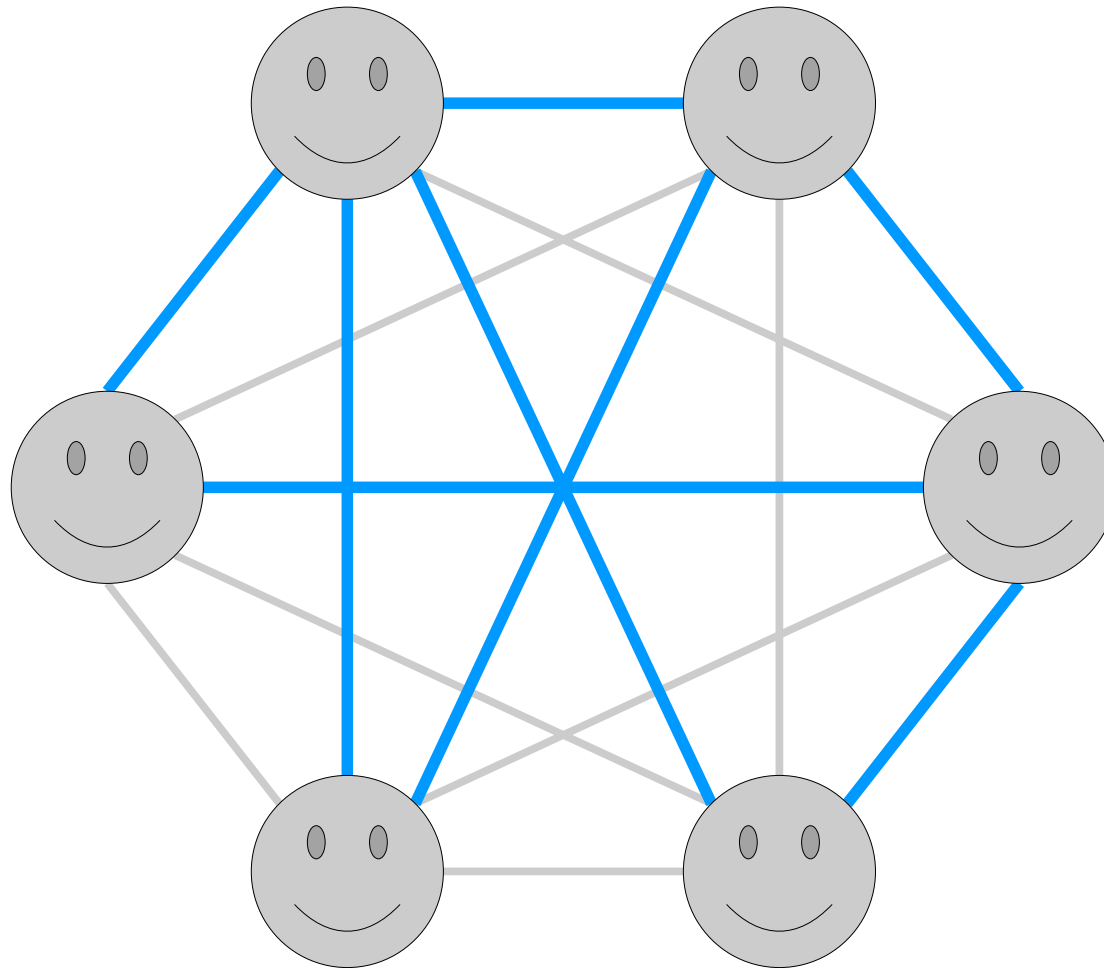


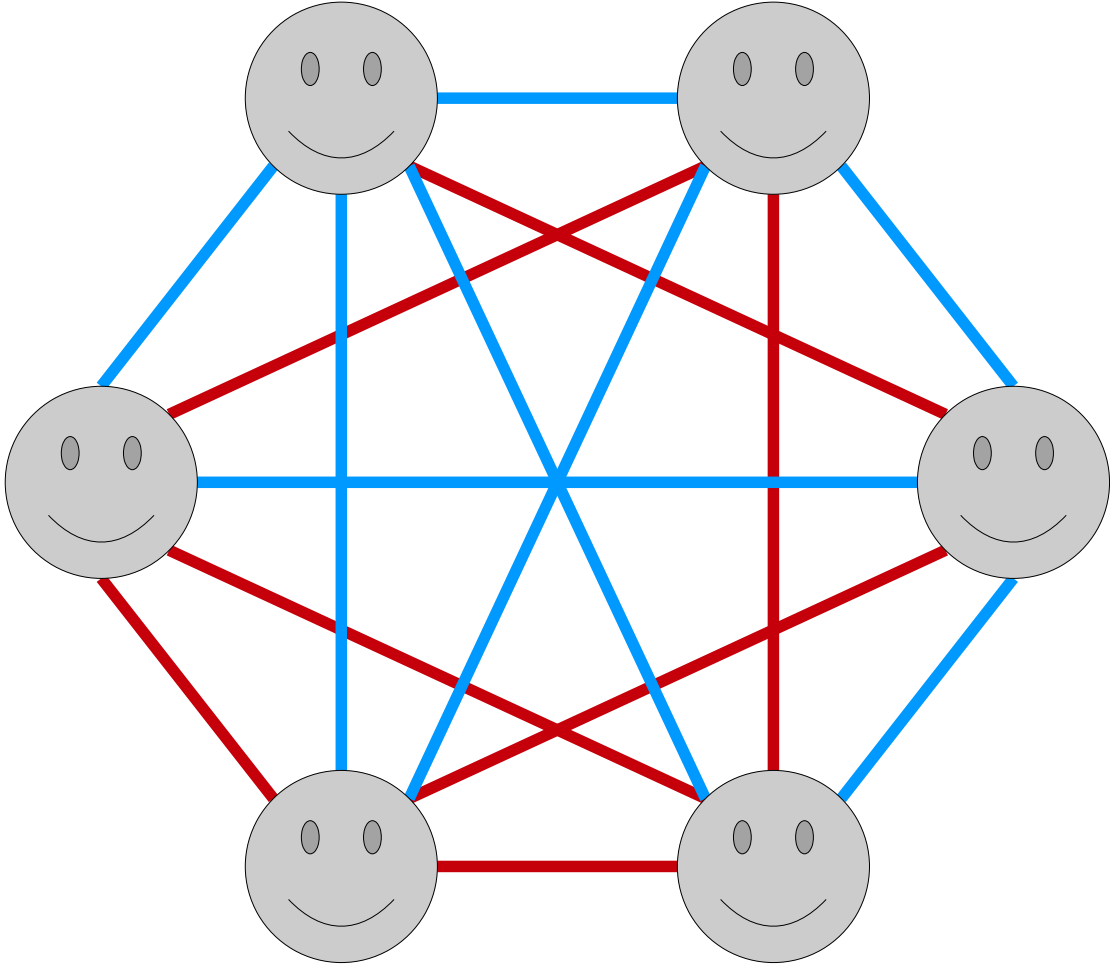


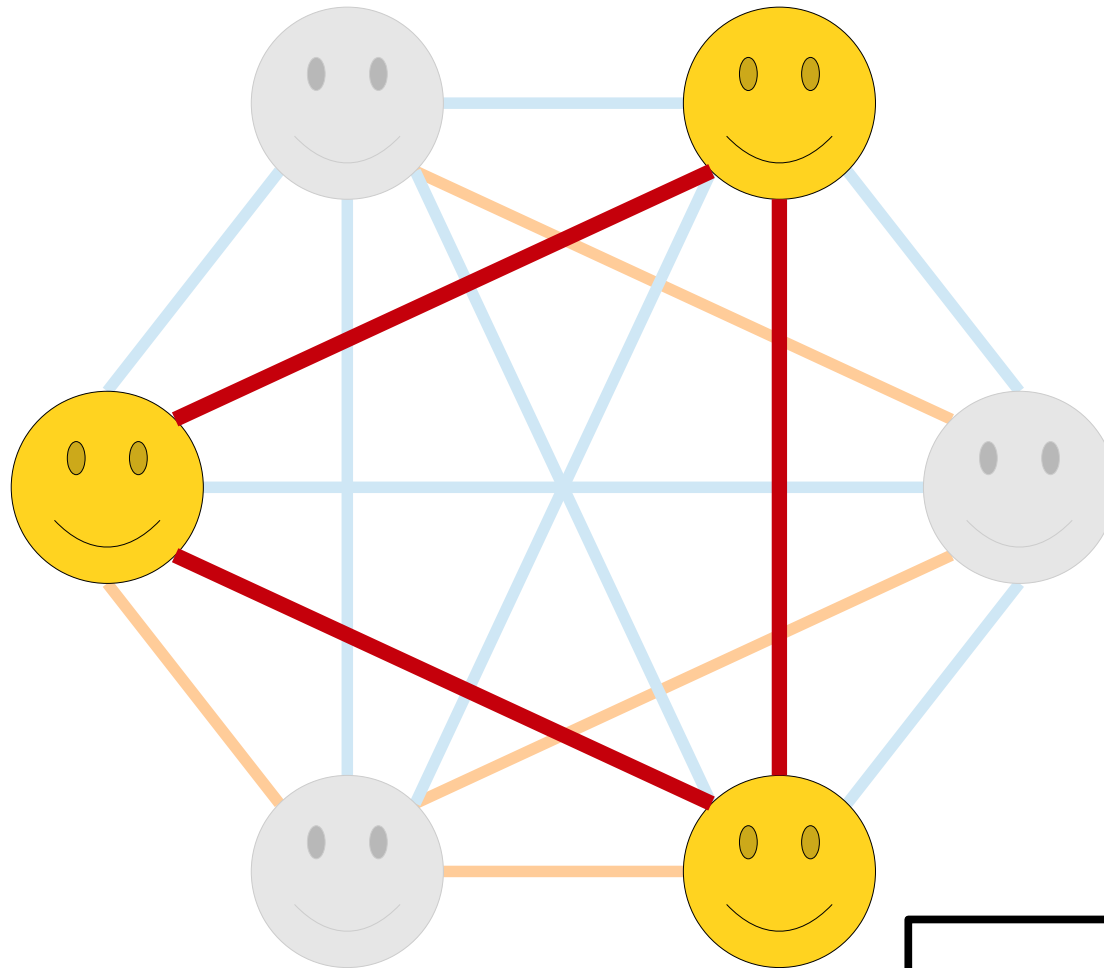




This graph is called K_6 , the **complete graph of order 6**. More generally, the graph K_n consists of n mutually adjacent nodes.







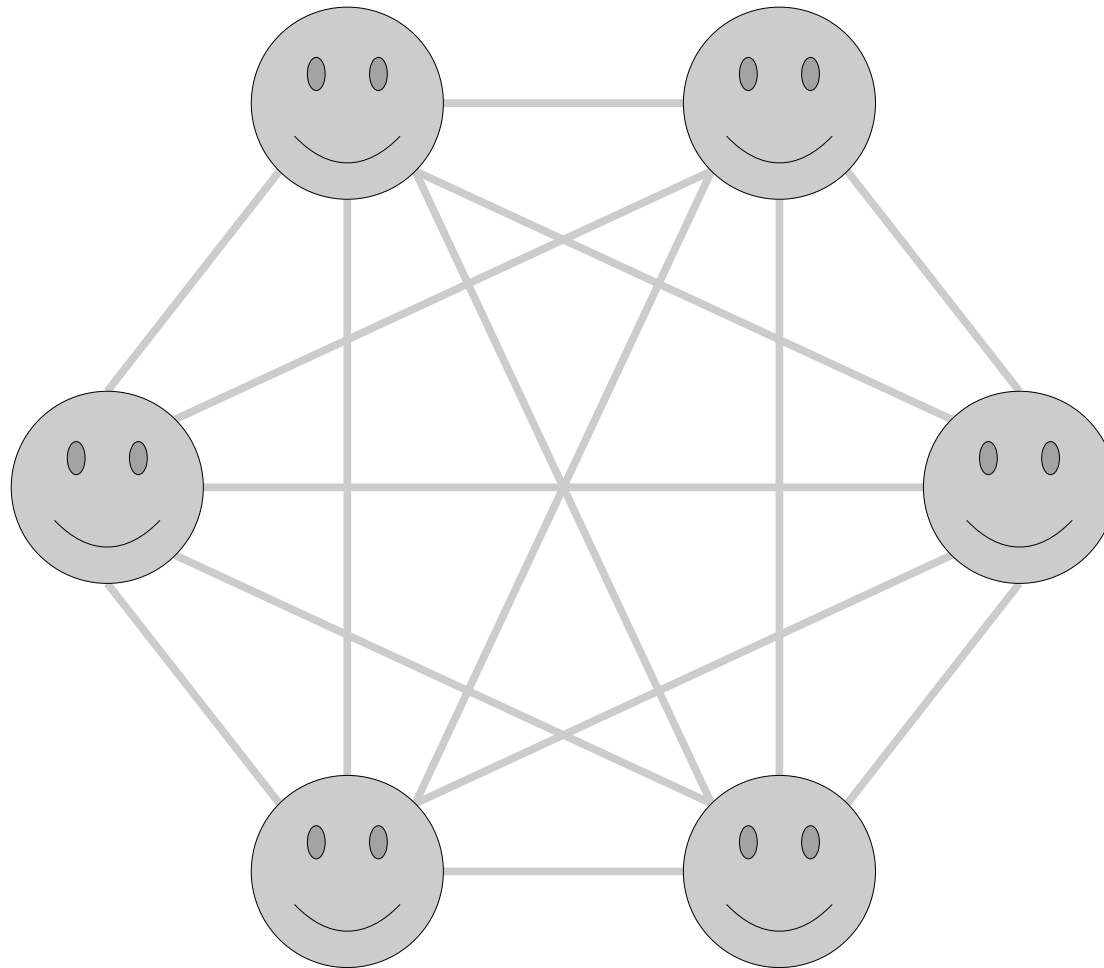
This is a
monochrome (one-
color) copy of K_3 .

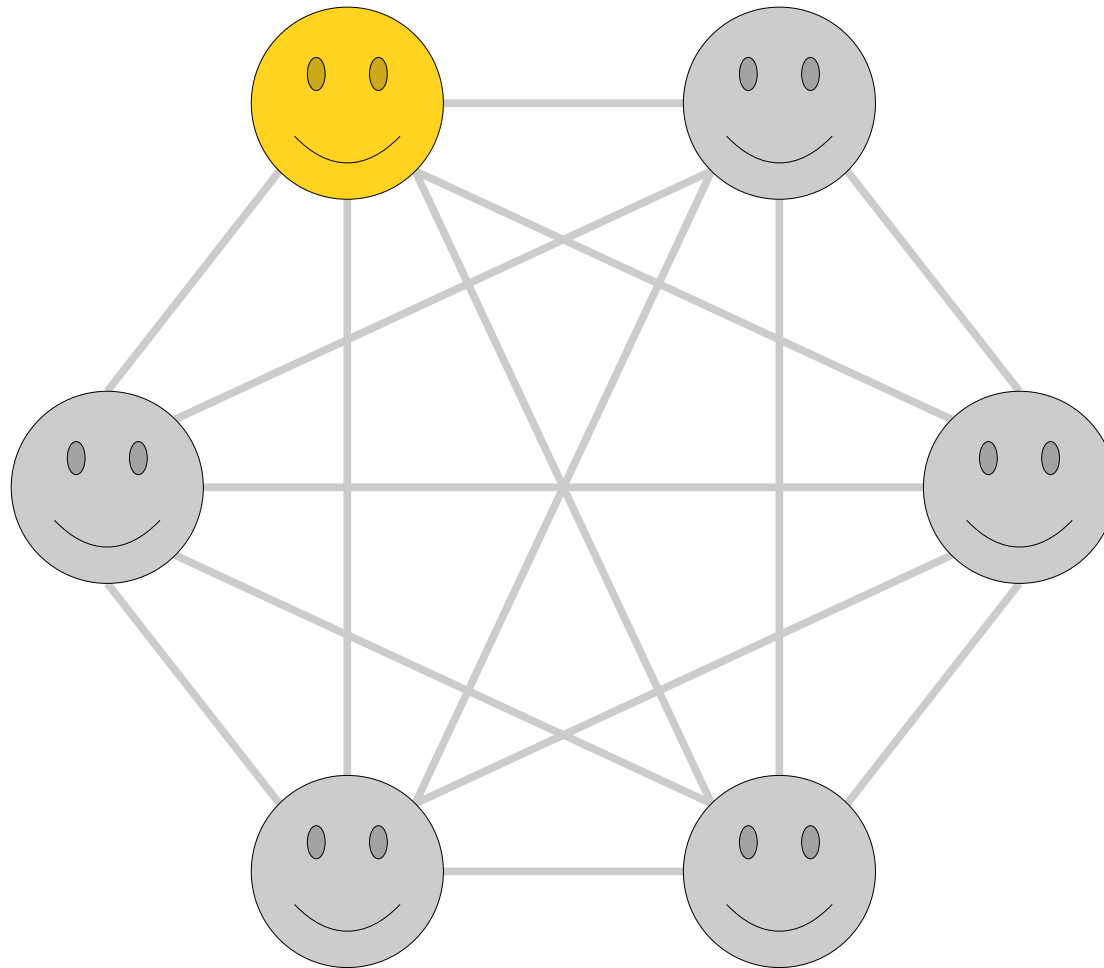
Friends and Strangers Restated

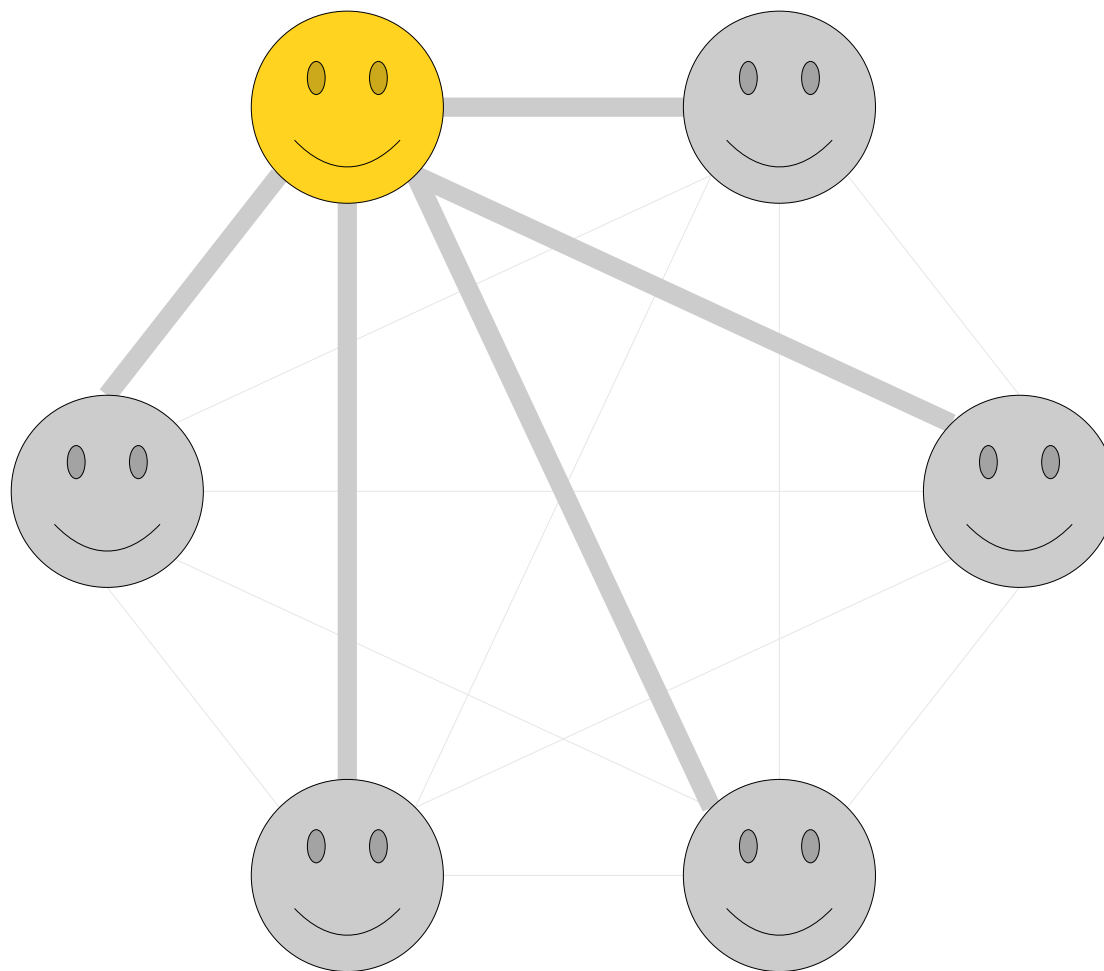
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

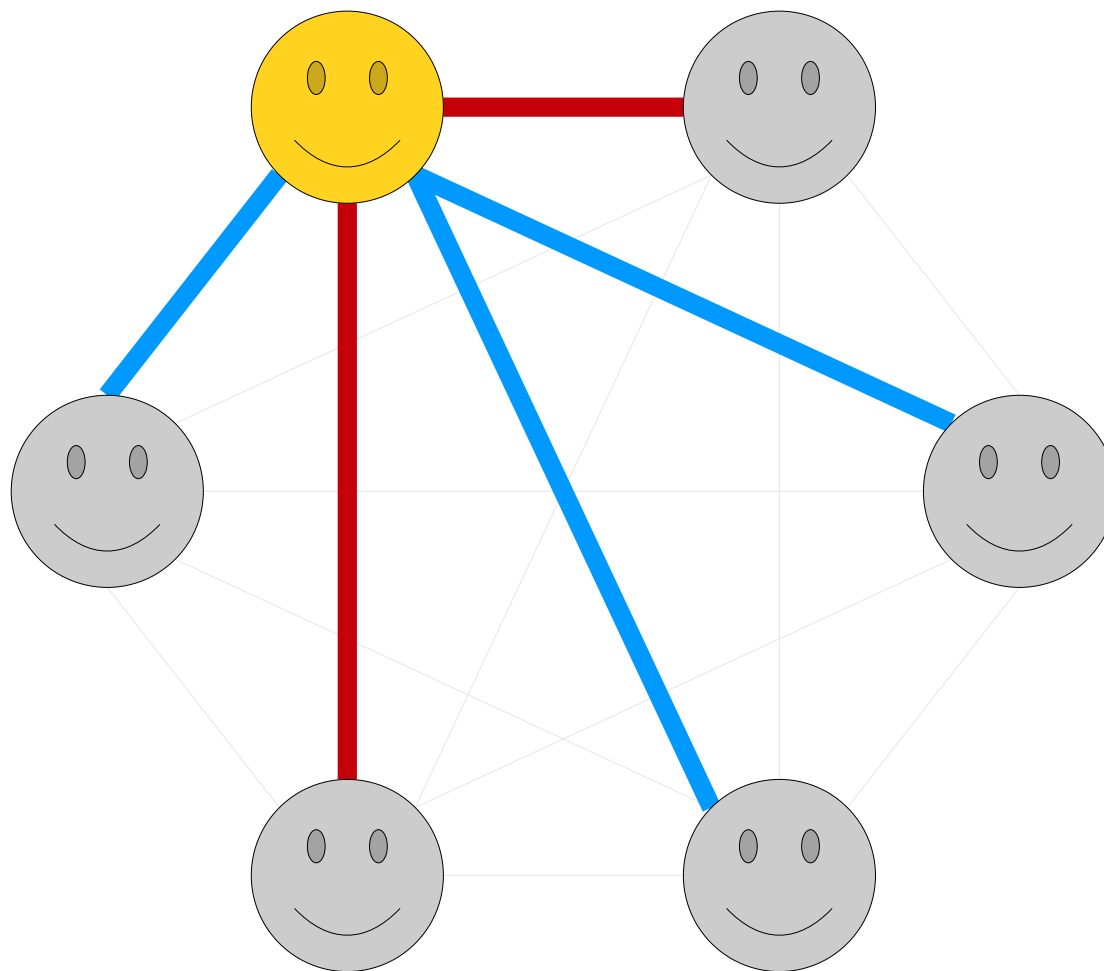
Theorem: Color every edge of K_6 either red or blue. The resulting graph always contains a monochrome copy of K_3 .

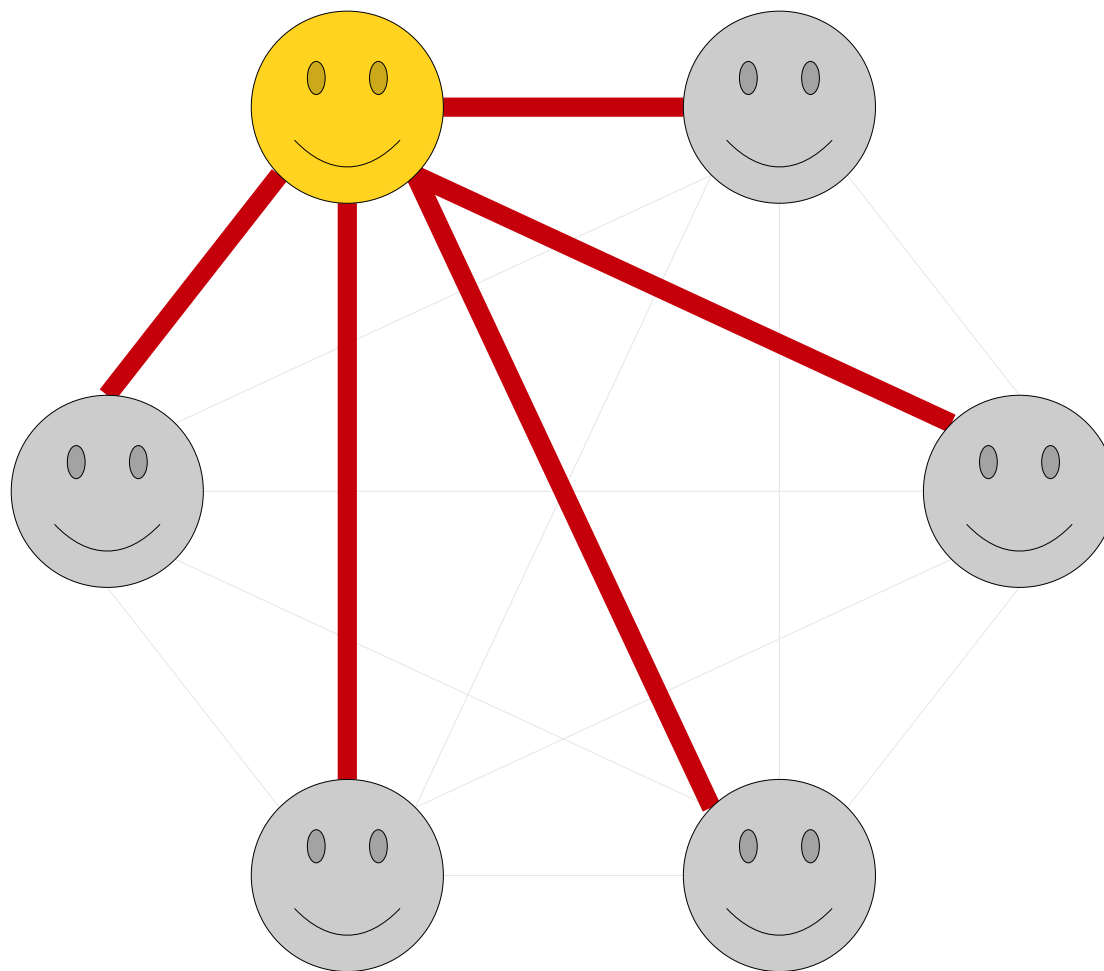
- How can we prove this?

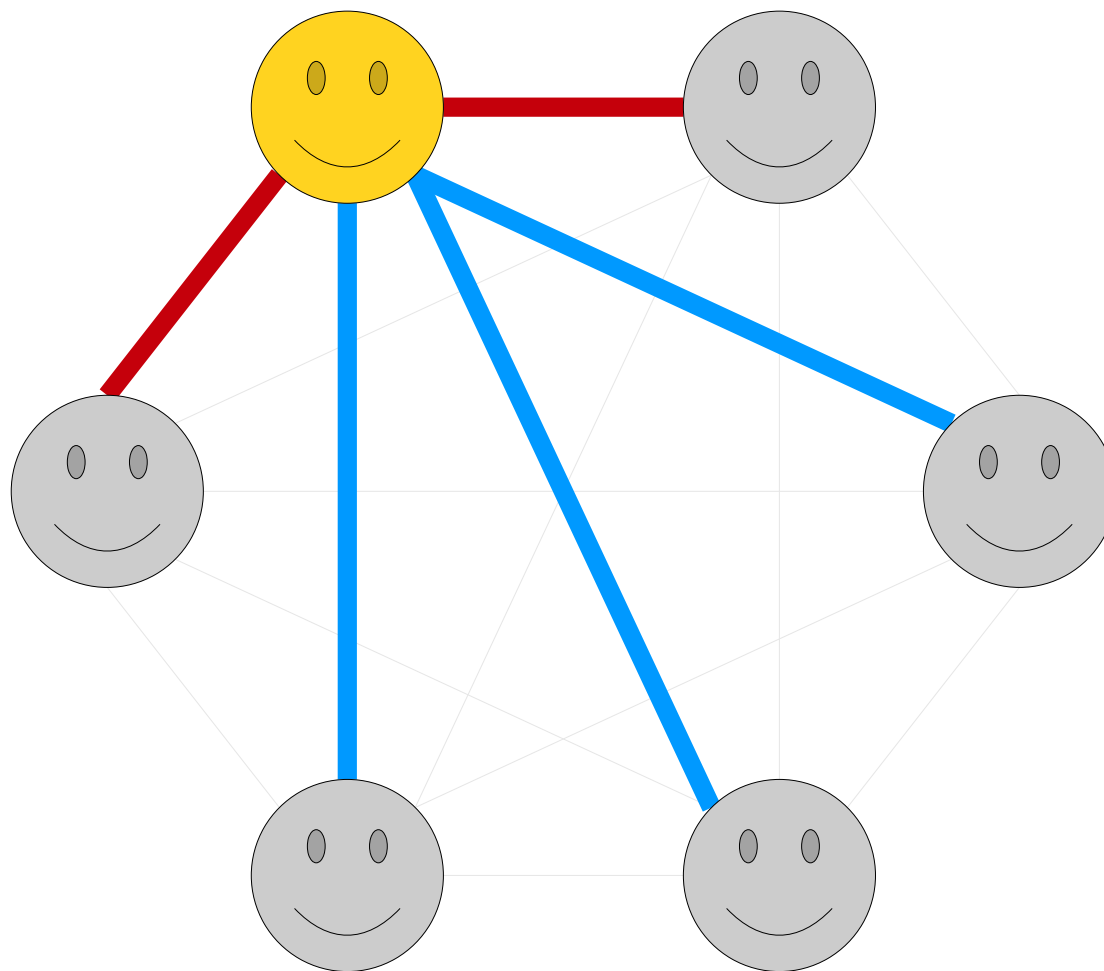


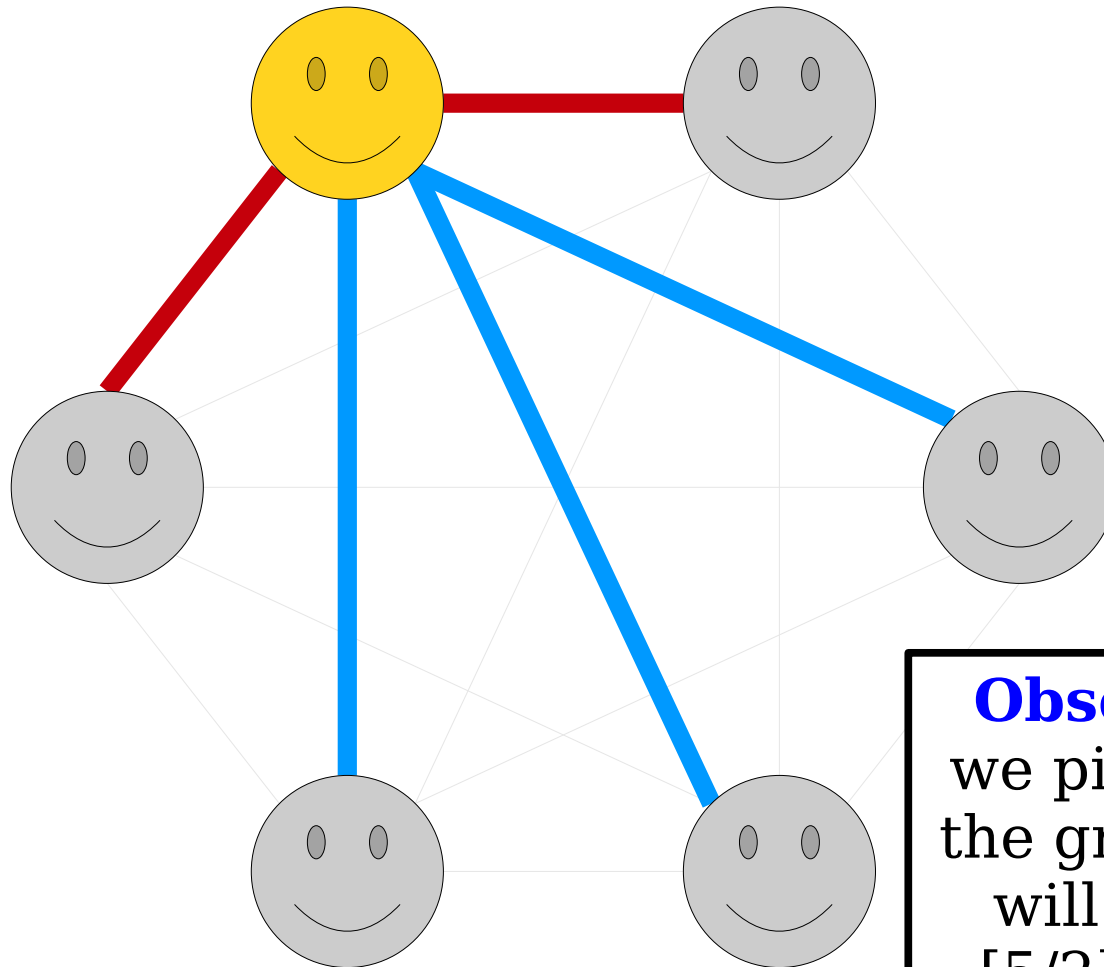




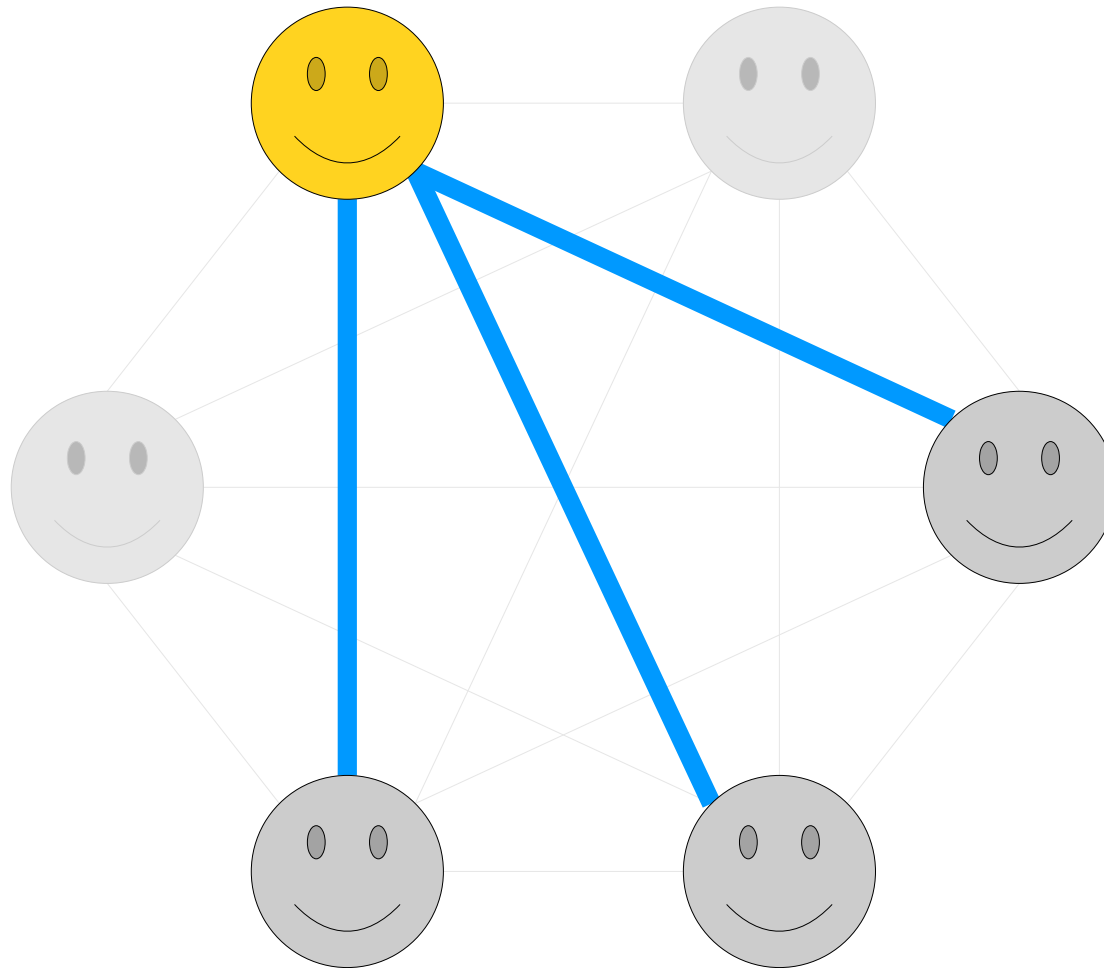


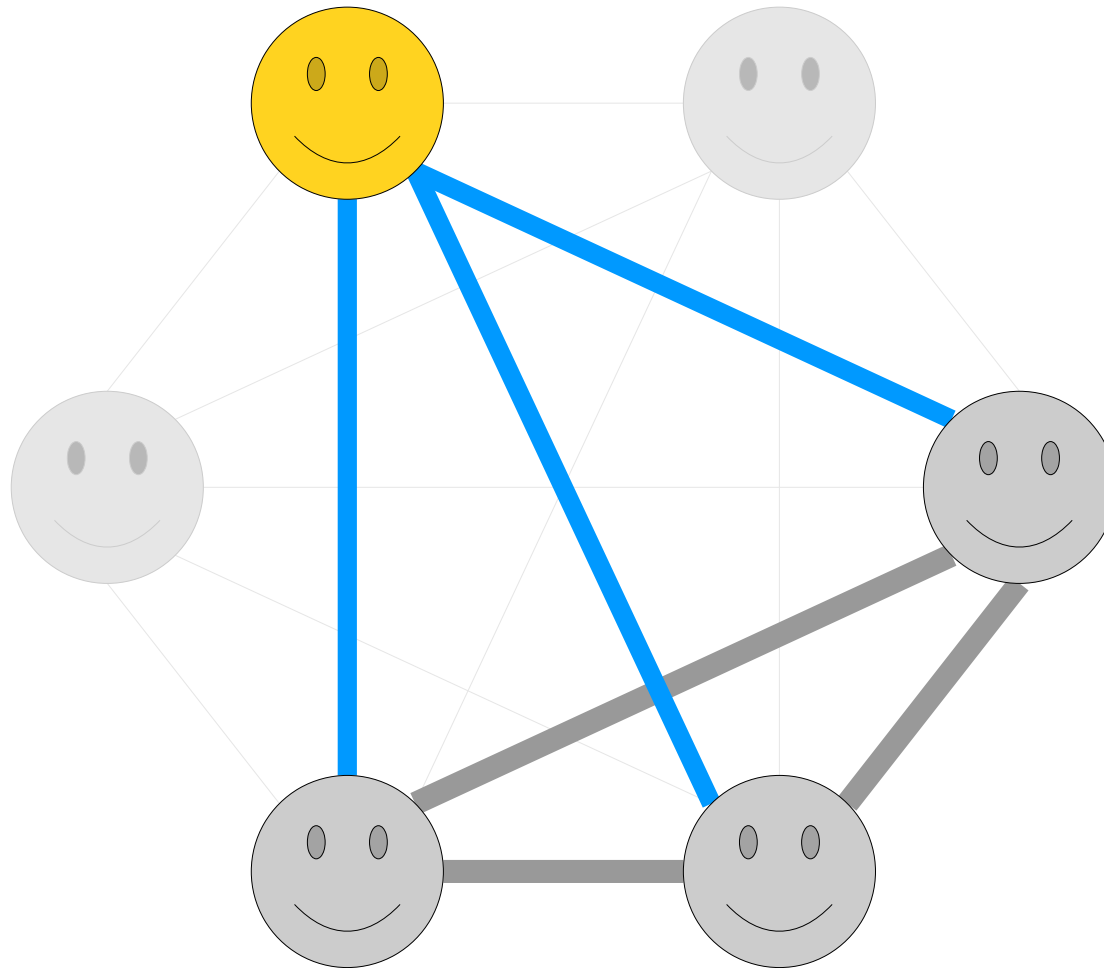


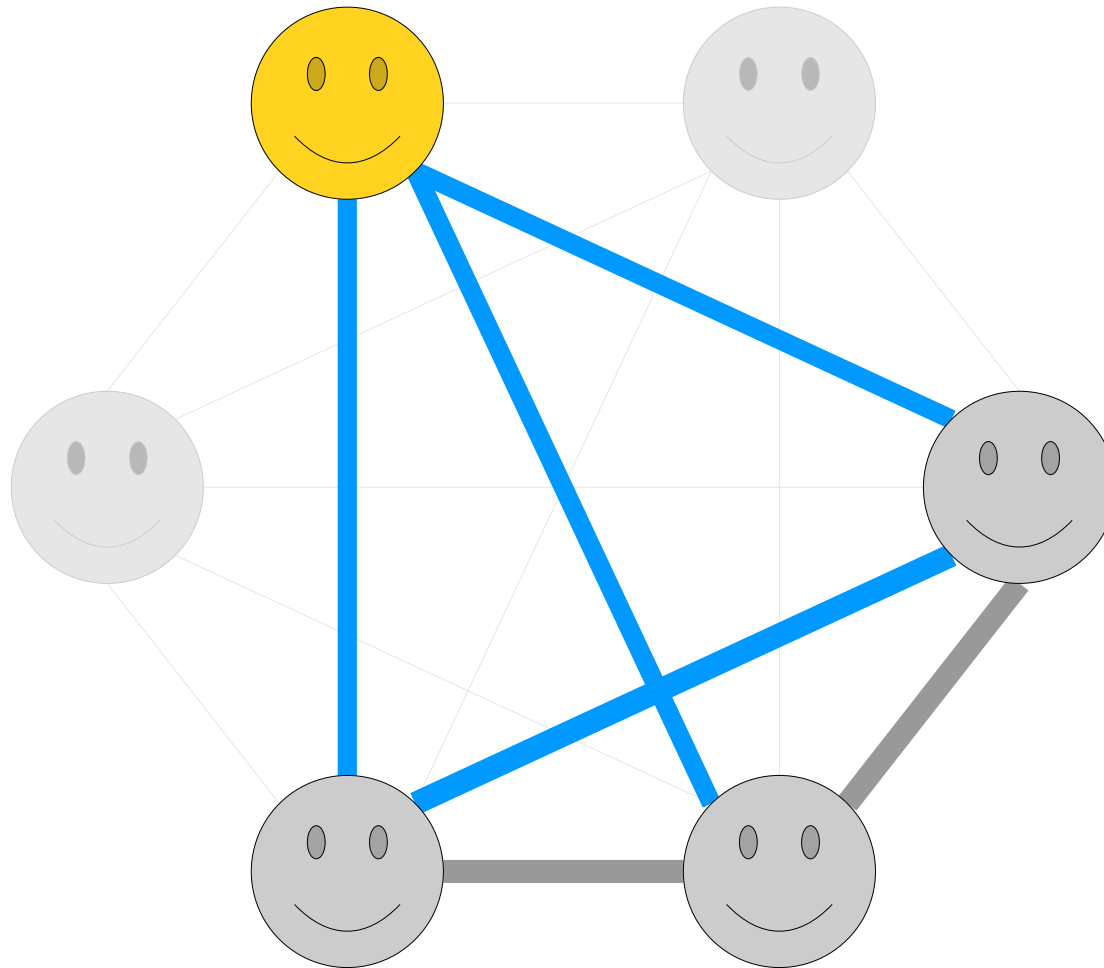


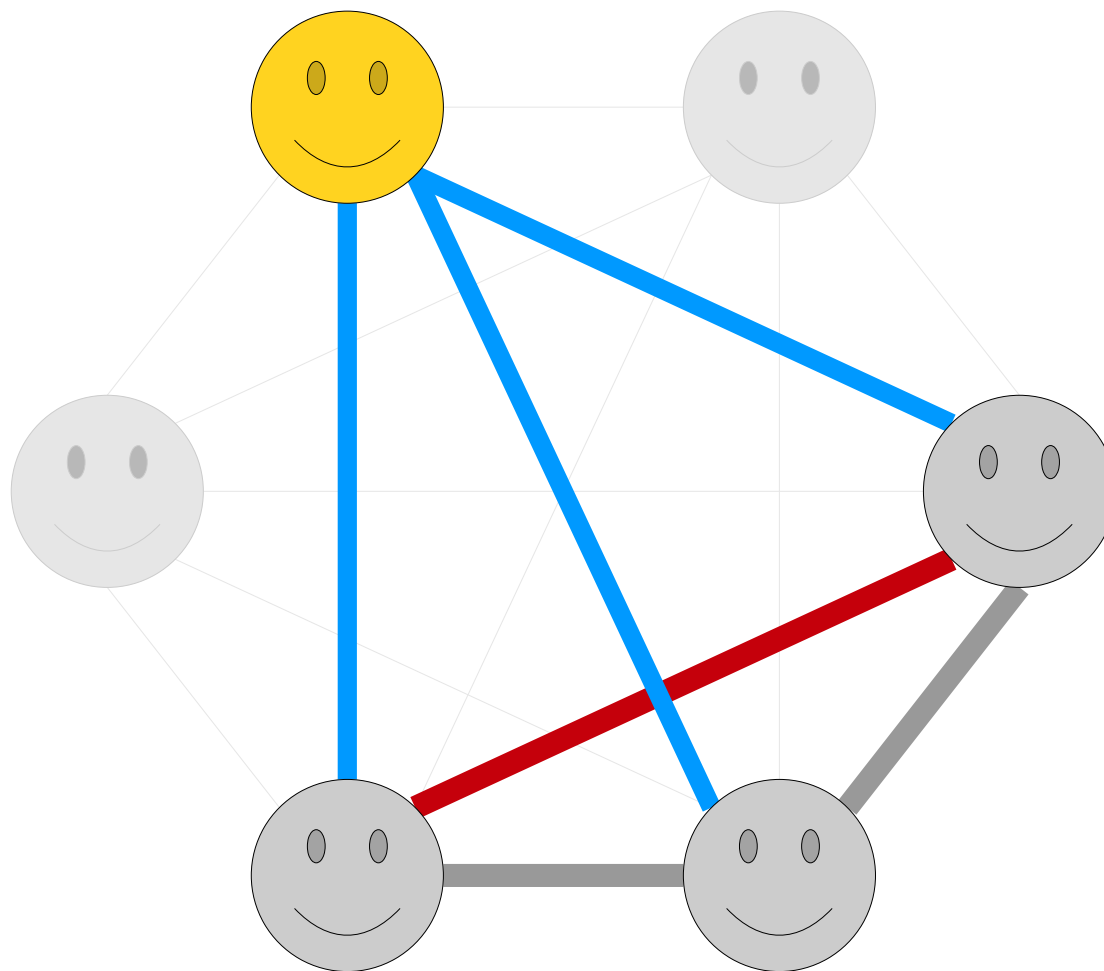


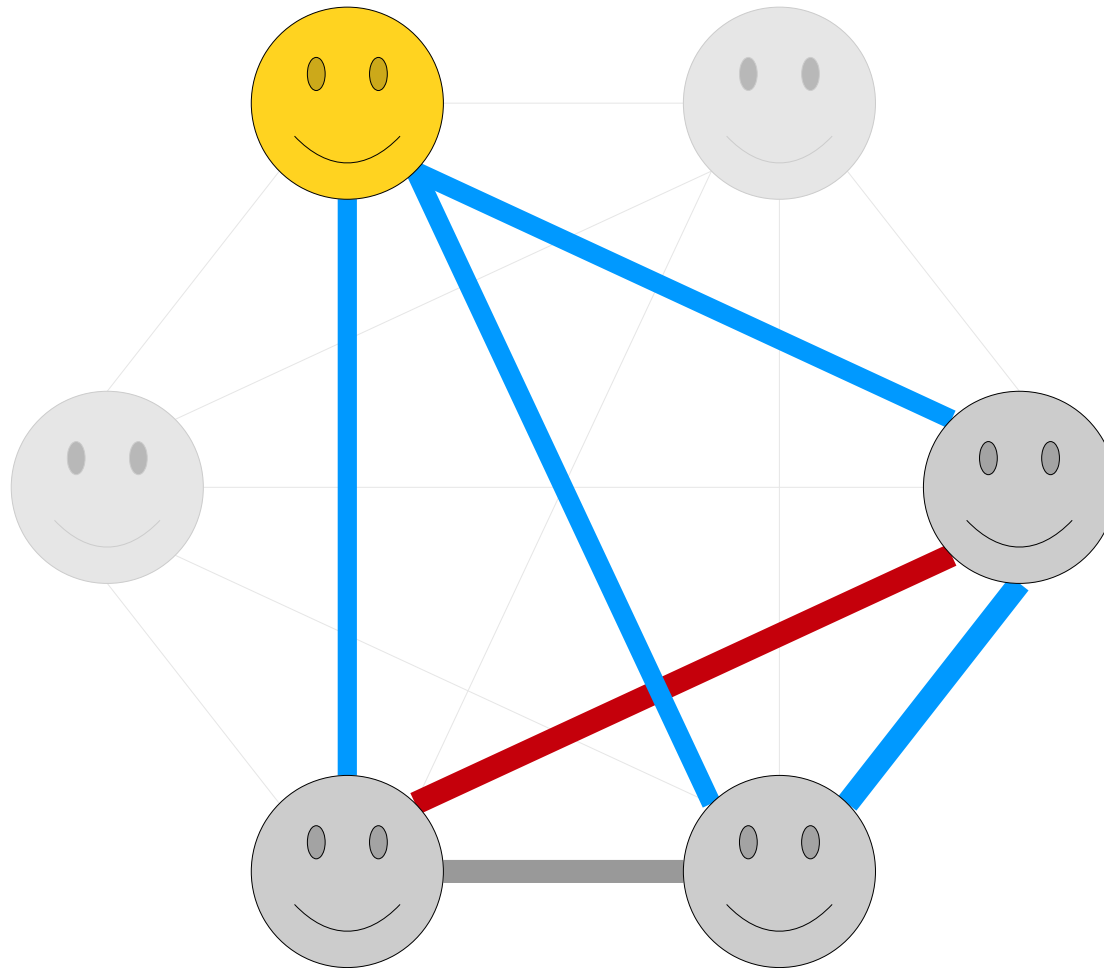
Observation 1: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

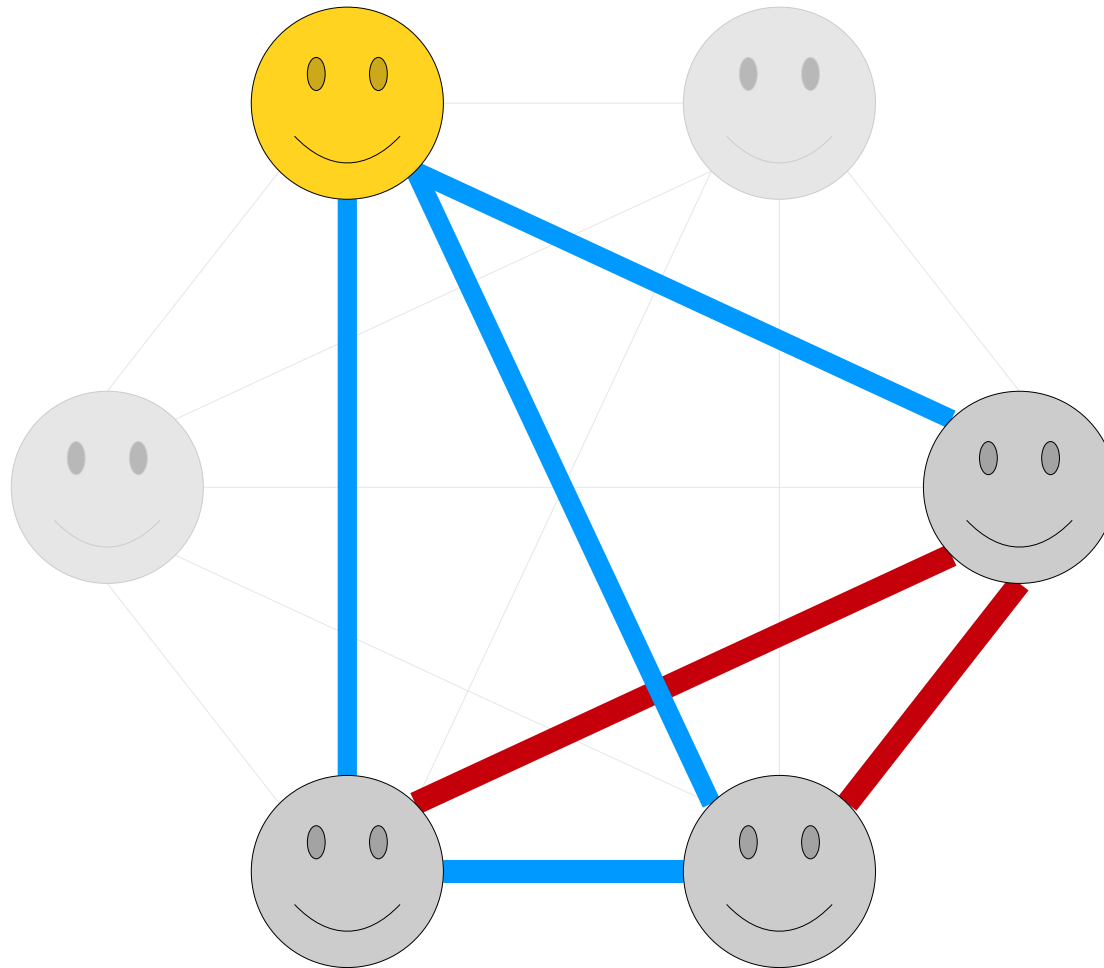


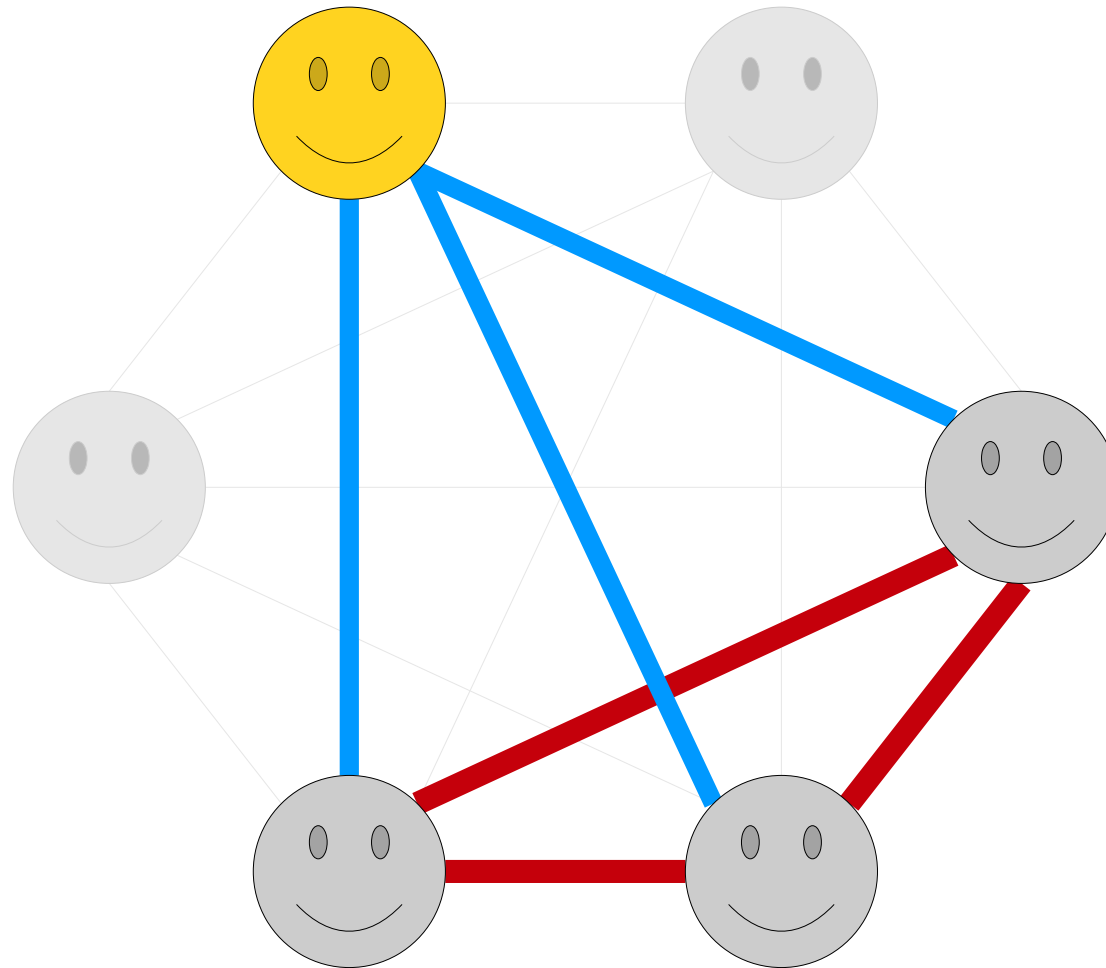












Theorem: Color each edge of K_6 red or blue. The resulting graph contains a monochrome copy of K_3 .

Proof: We need to show that the colored K_6 contains a red copy of K_3 or a blue copy of K_3 .

Pick some node x from K_6 . It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Without loss of generality, assume those edges are blue.

Let r , s , and t be three of the nodes adjacent to node x along a blue edge. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are blue, then one of those edges plus the two edges connecting back to node x form a blue K_3 . Otherwise, all three of those edges are red, and they form a red K_3 . Overall, this gives a red K_3 or a blue K_3 , as required. ■

Ramsey Theory

- This proof is a special case of a broader family of results called ***Ramsey theory***.
- ***Theorem (Ramsey)***: For any natural number s , there is a number $R(s)$ such that
 - for all $n < R(s)$, there's a way to color the edges of K_n red and blue so there are no monochrome copies of K_s , and
 - for all $n \geq R(s)$, every way of coloring the edges of K_n red and blue always has a monochrome copy of K_s .
- Take Math 108 (combinatorics) to learn more!
- A more philosophical (and less literal) take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

Time-Out for Announcements!

Problem Sets

- Problem Set Three was due today at 1:00PM.
 - You can use a late day to extend the deadline to Saturday at 1:00PM if you'd like.
- Problem Set Four goes out today. It's due next Friday at 1:00PM.
 - It's all about graphs and graph theory, and you'll see some really cool results!
 - Because the midterm is on Tuesday, we've made this problem set shorter than the previous problem sets.

Midterm Logistics

- Our first midterm is next Tuesday from 7PM – 10PM.
- Best of luck on the exam – ***you can do this!*** We're all cheering you on.
- There's plenty of extra problems online if you're looking to get some additional practice.
- Thanks to Benson for running a review session yesterday! The video is available online.
- Feel free to ask questions on EdStem over the weekend.
- There is no lecture on Monday.

My Advice

- **Do** block out some dedicated time to work through practice problems.
- **Do** get the TAs to review your answers to those problems; ask privately on EdStem.
- **Do** take some time this weekend to take a walk, smell the rosemary bushes on campus, and watch the bees buzz.
- **Don't** pull an all-nighter studying for the exam.
- **Don't** skip meals or alter your daily routine to fit in time for studying.
- **Don't** panic. You can do this!

Back to CS103!

A Little Math Puzzle

“In a group of $n > 0$ people ...

- 90% of those people enjoyed *CODA*,
- 80% of those people enjoyed *Nomadland*,
- 70% of those people enjoyed *Parasite*, and
- 60% of those people enjoyed *Knives Out*.

No one enjoyed all four movies. How many people enjoyed at least one of *CODA* and *Parasite*?”

Other Pigeonhole-Type Results

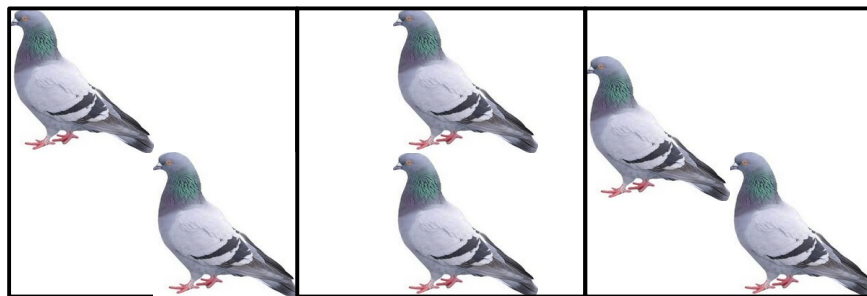
*If m objects are distributed into n boxes, then **[condition]** holds.*

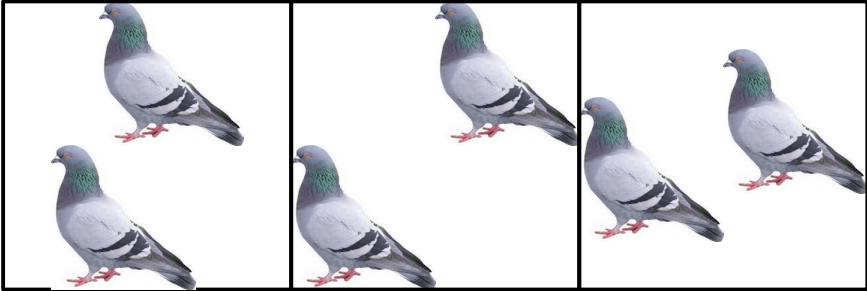
*If m objects are distributed into n boxes, then **some box is loaded to at least the average m/n , and some box is loaded to at most the average m/n .***

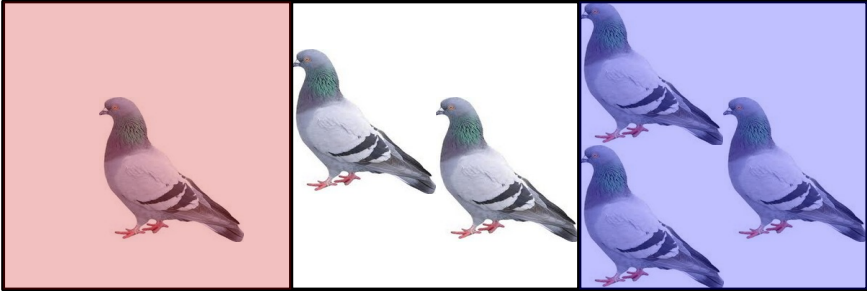
*If m objects are distributed into n boxes, then **[condition]** holds.*



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Theorem: If m objects are distributed into n bins, then there is a bin containing more than m/n objects if and only if there is a bin containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

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For simplicity, denote by x_i the number of objects in bin i .

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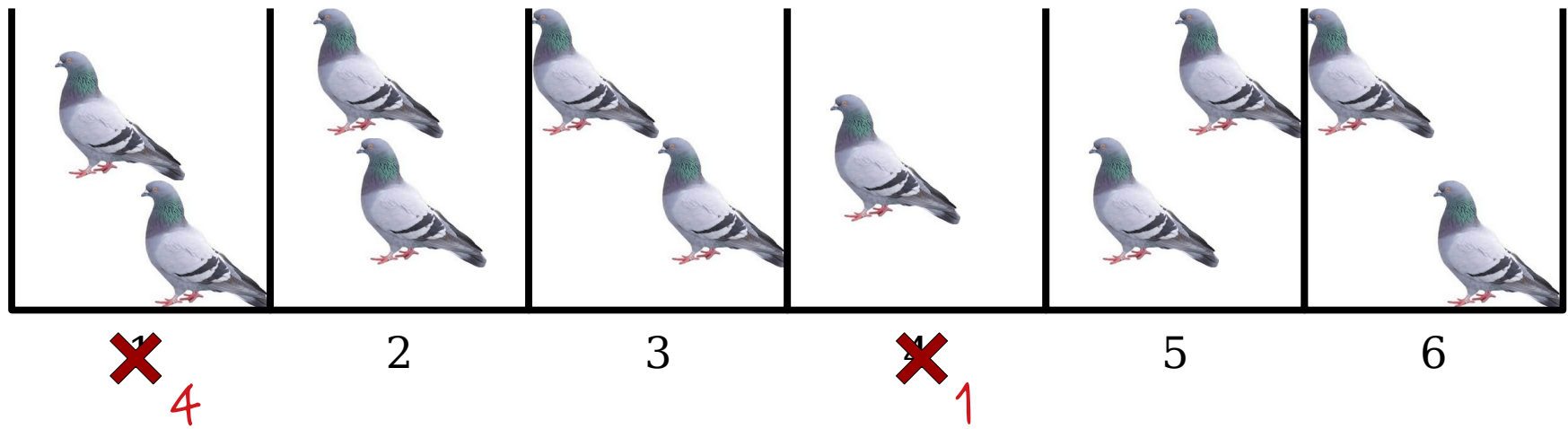
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This magic phrase means "we get to pick how we're labeling things anyway, so if it doesn't work out, just relabel things."



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$$m = x_1 + x_2 + x_3 + \dots + x_n$$

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This third step follows because each remaining bin has at most m/n objects. Grouping the n copies of the m/n term here tells us that

$$\begin{aligned} m &< m/n + m/n + m/n + \dots + m/n \\ &= m. \end{aligned}$$

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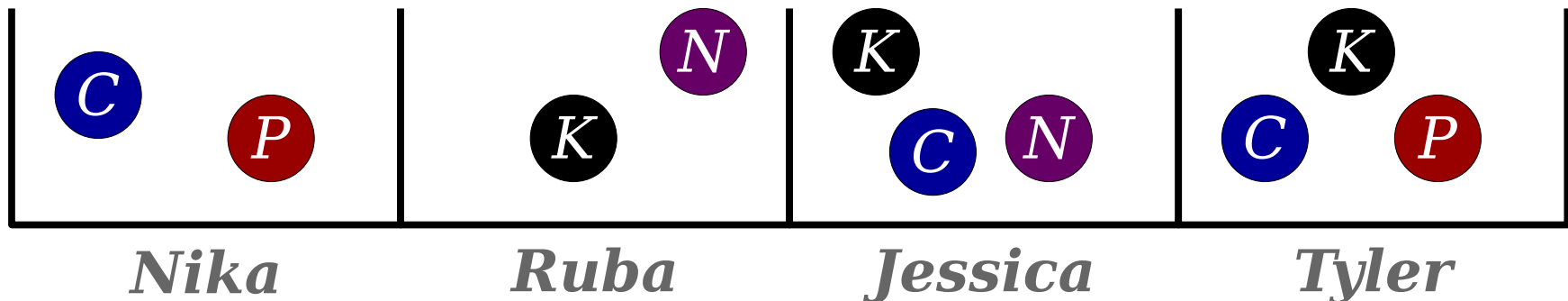
“In a group of $n > 0$ people ...

- 90% of those people enjoyed **CODA**,
- 80% of those people enjoyed **Nomadland**,
- 70% of those people enjoyed **Parasite**, and
- 60% of those people enjoyed **Knives Out**.

No one enjoyed all four movies. How many people enjoyed at least one of *CODA* and *Parasite*?”

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



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$$\begin{aligned} & .9n + .8n + .7n + .6n \\ & = 3n \end{aligned}$$

Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

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Insight 5: No one enjoyed more than three movies...

Insight 6: ... so no one enjoyed fewer than three movies ...

Insight 7: ... so everyone enjoyed exactly three movies.

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Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: Everyone liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *CODA* and *Parasite*.

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Theorem: In the scenario described here, all n people enjoyed at least one of *CODA* and *Parasite*.

Proof: Suppose there is a group of n people meeting these criteria.

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$$.9n + .8n + .7n + .6n = 3n,$$

and since there are n people, there are n bins.

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Now suppose for the sake of contradiction that someone didn't enjoy *CODA* and didn't enjoy *Parasite*.

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We've reached a contradiction, so our assumption was wrong and each person enjoyed at least one of *CODA* and *Parasite*. ■

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No one enjoyed all four movies. How many people enjoyed at least one of *CODA* and *Parasite*?”

Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
 - There is always a way to fairly split rent among multiple people, even if different people want different rooms. (*Sperner's lemma*)
 - You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. (*Mountain-climbing theorem*)
 - If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. (*Brower's fixed-point theorem*)
 - A complex process that doesn't parallelize well must contain a large serial subprocess. (*Mirksy's theorem*)
 - Any positive integer n has a nonzero multiple that can be written purely using the digits 1 and 0. (*Doesn't have a name, but still cool!*)

More to Explore

- Interested in more about graphs and the pigeonhole principle? Check out...
 - ... **Math 107** (Graph Theory), a deep dive into graph theory.
 - ... **Math 108** (Combinatorics), which explores a bunch of results pertaining to graphs and counting things.
 - ... **CS161** (Algorithms), which explores algorithms for computing important properties of graphs.
 - ... **CS224W** (Deep Learning on Graphs), which uses a mix of mathematical and statistical techniques to explore graphs.
- Happy to chat about this in person if you'd like.

Next Time

- ***No class on Monday - good luck on the midterm!***
- ***Then, when we get back:***
 - ***Mathematical Induction***
 - Reasoning about stepwise processes!
 - ***Applications of Induction***
 - To numbers!
 - To anticounterfeiting!
 - To puzzles!